Written on the Wall

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This list contains more than half of Graffiti's conjectures from the original version of "Written on the Wall" and some newer ones, but eventually I hope to include here most or all of them. Temporarily, some facts may be lost or even distorted, because of the difficulties with copying files from my old Macintosh; some information in the old files was expressed by the character of fonts and it for all practical purposes unretrievable. I'll be grateful for all corrections, missing information and of course any information on new developments.

G always denotes a graph, n the number of its vertices and eigenvalues (unless it is explicitly stated) denote eigenvalues of the adjacency matrix of G.

9. 96 Chronology of comments is often in conflict with their position in the most relevant part of the text and I am trying to solve this problem by using dates as quotation marks. 9. 96.

Starting with 747 all conjectures are about connected graphs, unless it is explicitly stated. Some conjectures are separated from the others by

$* * *$

to indicate that I knew the answer before they were included here. All of the conjectures listed below, with exception of 117, 118 and 119 were made by Graffiti, contrary to occasional reports that I was testing my own conjectures with the program.

4. 98. Mekkia Kouider and Peter Winkler made recently some incorrect statements about the program, but they are retracted in conjecture 127. 4. 98.

7. 98. The original version of this list contained a request for easily computable graph invariants. I would still appreciate any such suggestions, and comments to 127 include a stronger version of the same. It is a "challenge" that the program can make an interesting conjecture about **any** *such invariant. This is not an ego trip - 127 explains why I make this challenge. 7. 98.*

Various follow-up questions are mostly my own and some were suggested by readers of this list. Please let me know if you notice any errors or other nonsense. I'll be very grateful for this. The numbers in round parentheses refer to related conjectures or those containing needed definitions. Eventually there will be more of them.

Some of my follow-up comments and questions may be far-fetched and other are very simple. They are mostly memos to myself, particularly when they are sketches of proofs, and they are often included here simply because this is much simpler than to keep a separate set of notes. These fragments may contain errors; I tried commenting memos out, but because of the size of this list they would be as good as lost.

All conjectures are restricted to graphs in which all involved concepts are well defined, so for example conjectures involving distance are only for connected graphs. Conjectures starting with numbers about 700 were made by versions of the program developed jointly with Ermelinda DeLaVina. Development of this version of the program was partially supported by grant 003652085-ARP.

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Updated May 98.

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* * *

-1. **Temperature** *of a vertex v of graph G is $d/(n-d)$, where d is the degree of the vertex v and n is the number of vertices of G.*

Conjecture. *The chromatic number of every graph is $\geq$ 1 + the average temperature of its vertices.*

Proof: Let us color vertices of G with k colors, so that adjacent vertices have different colors. Let G^* be the graph obtained from G by adding all edges joining vertices of different color. If k is the chromatic number of G, then the chromatic number of G^* is also k. A simple computation shows that the average temperature of vertices is exactly is $k-1$. Clearly the temperature of every vertex of G^* is $\geq$ than its temperature in G, and in particular the equality holds true iff G is complete k-partite.

10. 96. **Ermelinda DeLaVina** *noticed that the construction in the proof consist of the blue edges in the sense of chromatic Ramseyan properties, see conjectures starting with 822, and that the final graphs are critical, (but not all critical graphs) with respect to our theorem in [DF].*

Actually I noticed already that the blue coloration, but this time with respect to K_n-free Ramseyan properties, was implicit in solutions to 121 which deals with generalizations of Turan Theorem. 10. 96.

Later I found that the mean temperature is closely related to the Turan bound (I noticed it earlier, but see for example 797.)

The conjecture that the chromatic number is greater or equal to the mean temperature was the first correct conjecture of Graffiti, which I could not prove out of hand. Once Graffiti added the term +1 to the left side of the inequality, the proof become obvious. This happened later several times. If the program (or myself) could *find a stronger conjecture, the proof would often became obvious, see for example 797 or 872. The principle that the boldest the conjecture the better became the dominating factor of the Dalmatian version of Graffiti, [V], and it is also discussed in [ACC].*

[V] Siemion Fajtlowicz, On Conjectures of Graffiti, Part V. 7th International Quadrenial Conference on Graph Theory, Combinatorics and Applications, Vol 1, p. 367- 376.

[ACC] S. F. Conjectures about Self and Acceleration of Programs, in preparation.

* * *

June 1. 86.

0. *For every connected graph the radius is not more than its independence number. This conjecture was proved in [FW], [FMS1] and later in [F]. At about the same time slightly stronger result appeared in [ESS].*

Radius is of course not a very good bound for the independence in most graphs, but rereading the proof of **Favaron** *it occured to me that the following algorithm may be of interest: select a vertex v in each component and partition vertices of this component into E and D depending on the parity of their distance from v. Then proceed recursively (actually at this stage one can use any other algorithm) to produce independent sets in E and D and then find a maximum independent set in the bipartite graph induced by these two independent sets and the "cross-edges" between E and D. I will refer to this algorithm as cascade.*

There are many interesting questions one can ask about the performance of cascade, for example, what is the worst and expected running time of the algorithm and how large independent sets will it generate compared to other algorithms? Of *particular interest is whether cascade will always generate an independent set as large as the average distance. (comp 2.) There must be quite a few other conjectures on this list which can be "cascaded" similarly as this one. 3. 90.*

8. 97. Since then I occasionally tried "to cascade" other conjectures (though not this one) for the independence number, with variable luck. The idea seem to work very well with 750. 8. 97.

[F] S. Fajtlowicz, A Characterization of Radius-Critical Graphs, JGT, 12(88), p. 529-532.

[ESS] P. Erdos, M. Sachs and V. Sos, Maximum Induced Trees in Graphs, JCT, 41(86) p.61-79.

1. *Chromatic number of a graph is not more than its rank +1.*

The rank of a graph is the rank of its adjacency matrix.

Francois Jeager told me (about a year) later that a somewhat stronger conjecture was made earlier by Cyriel Van Neufallen, [CDS].

9. 97. Graffiti made number of times another conjecture listed in [CDS] namely that the chromatic number of a graph is $\leq$ 1 + the number of non-positive eigenvalues. This conjecture was made by Amin and Hakimi. I checked with Professor Cvetkovic and as far as he knows, the conjecture is still open. This conjecture may be irrelevant to the communication complexity problem of Lovasz discussed below (it is rank which is significant) but it seems to me (and the Dalmatian version very clearly shares this opinion) that it is a very good conjecture about the chromatic number. 9. 97.

(*) Conjecture 54 implies that a d-regular counterexample to the conjecture must have at most d(d-1) vertices.

Mark Ellingham, Vanderbilt University *told me that, using a computer he verified the conjecture for all graphs of chromatic number at most 7 or rank at most 8, [EL]. He also told me that the same was done independently by* **Gordon Royle, University of Western Australia** *for all graphs of chromatic number at most 8 or rank at most 9.*

(**) **Noga Alon, Tel–Aviv University** *and* **Paul Seymour, Bell Communication Research** *found a 64 vertex graph with chromatic number 32 and rank 29. They ask if there is a polynomial bound for chromatic number in terms of the rank. The question is motivated by a result of Lovasz that a positive answer would solve an interesting problem in complexity of communication. 7.88.*

The **communication complexity***, [L], (folowing a model proposed by Yao) of a* $0-1$ *matrix* C *is the minimum number of rounds needed to partition it into constant (i.e each entry is* 0 *or each entry is* 1*) submatrices, where in each round one can split current submatrices into two either vertically or horizontally. Let* r *be the rank of* C *and* κ *its commumnication complexity. Since the above subdivisions may not decrease the rank by a factor greater than 2, it follows that* $\kappa(C) \geq log r$*, [MS]. On the other hand* $\kappa(C)$ *is bounded by the rank of* C*. Communication complexity problems occur in design of VLSI's, [L]*

Alexander Razborov, Steklov Institute, Moscow, *proved that chromatic number can not be bounded by a linear function of rank, [R].*

The **non-deterministic communication complexity** *is the minimum number of all - 1 submatrices covering all 1's of the communication matrix* C*.* **Ran Raz, Princeton** *and* **Weizmann Institute***, and* **Boris Spieker, University of Twente, Netherlands** *have shown that the gap between the chromatic number* *and the rank can be super-polynomial (i.e., the chromatic number can be larger than any polynomial of the rank). Their examples are graphs whose vertices are perfect matchings of complete bipartite graphs* $K_{n,n}$*, two being adjacent iff they span a hamiltonian cycle. They found exact formula for the rank which is equal to* $\binom{2k}{k}$*, where* $k = n - 1$*, i.e., rank is* $2^{O(n)}$*. The non-determinsitic c.c. which is a lower bound for d.c.c. is* $\Omega(nloglogn)$*.*

In the section in which they compute the rank they use group representations along with Ferrer diagrams and they gallantly acknowledge there help of Lovasz and Conway. Perhaps one could display an explicit collection of matchings with (a hopefully simple) argument that it is a basis The simplicity of the formula for the rank suggests that there may be a simple combinatorial interpretation of bases. On the other hand the method used in the paper seems very general and should be helpful in computing ranks of other Cayley graphs, at least in the case of permutation groups; matchings in question are treated as permutations of S_n*. To use their method one probably have to assume that the set of generators is closed with respect to inner automorphisms, but that's still leaves plenty of interesting examples.*

The communication problem discussed by Raz and Spieker was proposed by Alok Aggarval, and since it led to such elegant mathematics the generalization of the problem discussed below is of interest, particularly that it is related to some long-open problems including existence of projective planes.

Suppose that edges of a complete graph on $n = 2k-1$ *vertices are colored with* n *colors and each of the colors is an input for one of* n *processors. The communication complexity problem is whether every two colors form a spanning path.*

The mentioned above related problem is whether the lattice L_n *with* 0*,* 1 *and* n *complementary elements can be represented as a lattice of partitions of* n*-element* *set. This is a variation of a (famous, before it was finally solved by Pavel Pudlak and Jiri Tuma) Birkhoff's problem whether every finite lattice can be represented as a lattice of partitions of a finite set. Some simple cases and variations of this problem are discussed in [EFM]; for example if* n *is even then it is a question of generating hamiltonian cycles (as in Raz-Spieker Theorem.) For prime and some other values of* n *the answer is "yes" by results of Kotzig. About a year or two ago, someone - but unfortunately I do not remember the name of the person - wrote to me that he was running (for a year or so) a program to find the answer for* n *equal twenty-something, which I guess gives a new meaning to this problem. The c.c. problem becomes the question of an algorithm for parallel machine to decide if* L_n *can be represented as a lattice of partitions of an* n*-element set.*

The result of Raz and Spieker seems to me an indication that the question must have positive answer, at least for all but finitely many n*. On the other hand if this conjecture is wrong then it is probably even more interesting.*

A version of the representation problem for $n = k^2$*, is equivalent to existence of projective planes of order* k*, [EFM]. 8.97.*

A much larger gap between the rank and communication complexity was established by **Noam Nisan and Avi Wigderson** *from* **The Hebrew University.** *They constructed 0-1 matrices with* $\kappa = \Omega(n)$ *and log of rank* $= O(n^{\alpha})$*, whith* $\alpha = log_3(2) = 0.63$*.. Using their method* **Eyal Kushilevitz** *from* **Technion, Haifa** *decreased later* α *to* $log_6(3) = 0.61$*.*

A boolean operation is **fully sensitive** *if its value at the zero vector is* 0 *and for any unit (i.e., Hamming weight 1) vector, the value is* 1*. The degree of a boolean operation is the degree of the unique multi-linear polynomial representing* *this operation. The construction of Nisan and Widgerson is based on existence of fully sensitive boolean n-ary operations of low degree ($n^{log_3(2)}$.)*

Nisan and Widgerson propose some further (weaker) conjectures, and to prove one of them it is enough to show that every boolean matrix of low rank contains a large constant submatrix. For graphs the following version of this problem is of interest: what is the best lower bound for the independence ratio (independence number/ the number of vertices) as a function of rank. The critical graphs of small rank should provide new conterexamples to the rank-coloring conjectures and though they will be almost certainly weeker than existing counterexamples they can make Graffiti to produce new related conjectures.

9. 97. This question can be also viewed as a version of Ramsey numbers: what is the smallest integer r(k,l) such that every graph with r vertices has rank l or more or contains a k-element independent; graphs of rank less than l can not contain cliques with l vertices so the existence of the numbers follows from Ramsey Theorem.

For example $r(n,2) = r(2,n) = n$, but $r(3,3) = 5$ because C_5 is the unique Ramsey-critical graph R(3,3). 9. 97.

The only counterexample examples to the original conjecture, which so far I was able to inform the program about, are derived from Alon-Seymour graph. Let G be the graph obtained from the n-dimensional cube by joining all pairs of vertices whose distances belong to a fixed set of numbers. Graphs of this form were among **Shearer's** *first counterexamples to some early conjectures. I noticed recently that the Alon-Seymour graph is of this form, but I do not know any other such examples. 8.97.*

Andrei Kotlov *and* **Laszlo Lovasz**, **Yale University** *proved that the chromatic number is* $O(2^{r/2})$*, where* r *is the rank, [KL]. Two vertices of a graph are* **twins** *if they have the same neighbors. Since removal of one of the twins does not affect neither rank nor the chromatic number, they consider only twin-free graphs and prove that these graphs can have at most* $O(2^{r/2})$ *vertices. Moreover they construct twin-free examples of this size. In the proof they use results of Kabatianski and Levenshtein whose methods (linear programming applied to sphere packing) are discussed in the 9th chapter of [CS].*

The bounds in terms of rank are further improved by **Kotlov** *in [K]. In both papers the crucial role is played by a largest subgraph of a rank smaller than* r*. Kotlov and Lovasz conjecture that the chromatic number is bounded by a subexponential function of rank. Kotlov has also lower bounds for the the independence number in terms of the rank (compare the above rank Ramsey-like problem) 9. 97.*

[AS] Noga Alon and Paul Seymour, A Counter-Example to the Rank-Coloring conjecture, JGT 88.

[CS] J. H. Conway and N. J. A. Sloane, Sphere Packings, Lattices and Groups, Springer - Verlag, Second Edition.

[EFM] A. Ehrenfeucht, V. Faber, S. Fajtlowicz and J. Mycielski, Representation of Finite Lattices as Partition Lattices of Finite Sets, Proc. UH Lattice Theory Conference, '73.

[EL] Mark Ellingham, Australasian Journal of Combinatorics in 1993.

[K] A. Kotlov, Rank and Chromatic Number of a Graph, JGT 1997, 1-8.

[KL] A. Kotlov and L. Lovasz, The rank and size of graphs, JGT, 1996, 185 - 189.

[L] Laszlo Lovasz, Communication Complexity, A Survey, Paths, Flows and VLSI layouts, technical report, H-1008, Eotvos U. and Princeton.

[MS] K. Mehlhorn and E.M. Schmidt, Las Vegas is Better than Determinism in VLSI and Distributed Computing, Proc. 14 ACM STOC 1982, pp. 322-349.

[NW] N. Nisan and A. Wigderson, On rank vs. Communication Complexity, Combinatorica 15(4) (1995) 557-565).

[R] Alexander Razborov, The Gap Between the Chromatic Number of Graph and its Rank is Superlinear, Discrete Mathematics 108 (92). pp 302-306.

[RS] Ran Raz and Boris Spieker, On the "log rank"-Conjecture in Communication Complexity", Combinatorica 15(4) (1995) 567-588)

2. *In every connected graph the average distance is not more than the independence number.*

W. Waller, UH *and myself proved that the average distance is not more than 1 + the independence number, [FW].*

The conjecture was proved by **Fan Chung, Bell Communication Research.** *If correct, conjecture 747 is a generalization of conjecture 2.*

Conjecture 69 contains a description of two algorithms, whose performance may be related to conj. 2.

P. Dankelmann, Lehrstuhl fur Mathematics, Aachen, Germany *found a sharp bound for the average distance in terms of the number of vertices and the independence number and characterized all critical graphs, solving a problem of Erdos stated in [FW]. 7.97.*

P. Firby and J.Haviland *from* **Exeter University, U. K.** *extended some of the above results to the kth power graphs i.e., graphs in which to vertices are adjacent if their distance in G is less than k. 7.97.*

Is there is a probability measure on the set of vertices of a connected graph such that randomly selected subset with the average distance of vertices is almost certainly independent. I don't think that there is such measure depending only on the degree sequence. The second conjecture may be related to problems discussed in 69. 8. 97.

[C] Fan Chung, The Average Distance is not more than the Independence Number, JGT, (88)

[D] P. Dankelmann, Average Distance and the Independence Number, Discrete Applied Mathematics, 51, 1994 73 - 83.

[FW] S. Fajtlowicz and W. Waller, On Two Conjectures of Graffiti, Congressus Numerantium, 55(86), p. 51 - 56.

[FH] P. Firby and J.Haviland, Independence and average distance in graphs, Discrete Applied Mathematics, 75(1997), p. 27-37.

[W] W. Waller, PhD Dissertation, UH.

3. *The weight of an edge with endpoints of degree x and y is the reciprocal of the square root of xy. The* **Randic index** *of a graph is the sum of weights of its edges. Conjecture: If G is a connected graph then the average distance between its distinct vertices is not more than its Randic index.*

The Randic index was proposed as a measure of "branching" of a tree and it is used by some chemists to predict the boiling point of certain hydrocarbons[KH]. In view of this conjecture it is interesting to remark that Wiener proposed for the same purpose an index which for simple graphs is the sum of all of its distances [KH], p.28.

comp conj's 12, 19, 20, 21, 27 and 28.

Gilles Caporossi and Pierre Hansen, University of Montreal, Canada *wrote system* **AutoGraphiX** *which among others searches for counter-examples, hints of proofs and refinements of conjectures. In their words the system has "aims similar some of those of "Graph" and to lesser extent to those of Graffiti, but pursued by other means." The latest version of "Graph", originally designed by Cvetkovic, is described in [CS]. The idea of AutographiX is to optimize an invariant subject to a class of constraints - ideally suited to search for counterexamples of many Graffiti's conjectures since they usually involve few invariants [GH]. On the basis of their experiments they conjecture that the difference between the Randic index and the average distance is at least* $\sqrt{n-1} - 2/n + 2$*, which would be sharp in stars. They list several other equally specific conjectures. See also 12, 15 and 16. 6. 97.*

[CDS] D. M. Cvetkovic, M. Doob and H. Sachs, Spectra of Graphs, Academic Press, 1980.

[CS] D. Cvetkovic and S. Simic, Graph theoretical results obtained by the support of the expert system "graph" Bull. de l'Academie Serbe des Sciences et des Arts, T CVII (1994) 19-41.

[GH] G. Caporossi and P. Hansen, Variable Neighborhood Search for Extremal Graphs, preprint 97.

[KH] L. B. Kier and L. H. Hall, Molecular Connectivity in Chemistry and Drug Research. Academic Press, 1976.

[R] M. Randic, J. Am. Chem. Soc. 97, 6609 (1975).

[W] H. Wiener, J. Am. Chem. Soc. 17, 2636 (1947).

4. *average distance is not more than the variance of the degree sequence + the sum of reciprocals of degrees.*

All statistical invariants like for example variance are derived from the degree sequence or other functions by treating them as random variables on uniform sample spaces.

An earlier conjecture that the average distance is not more than the sum of reciprocals of degrees was disproved **by Paul Erdos, Janos Pach and Joel Spencer, [EPS].**

Some results about variance of the degree sequence can be found in [B] and [CG], see conj. 3. 8. 97.

[B] F. K. Bell, A note on the irregularity of graphs, Linear Algebra and its Applications, 161 (1992) 45-54.

[EPS] Paul Erdos, Janos Pach and Joel Spencer, Ars Combinatoria, 1988.

5. *average distance is not more than mode of distance + sum of reciprocals of degrees.*

6. *average distance is not more than the variance of the degree sequence + maximal frequency of the degree sequence, (4.)*

The conjecture was refuted by **James B. Shearer, IBM Research Center at Yorktown Heights.** *10.89.*

7. *mode of distance is not more than radius + Randic index. (3, 9.)*

8. *mode of distance is not more than average distance + Randic index. (3, 9.)*

9. *Let m be the mode of distance of connected graph, i.e the distance which occurs most often. Conjecture: m is not more than the average distance + the matching number.*

This conjecture was refuted by **Hi Dong Qi, Dept. of Applied Mathematics, Beijing Institute of Technology***, [HE]. 3.90.*

10. *average temperature is not more than rank. (-1, 1.)*

This conjecture is correct. s.f 1.88.

11. *average temperature is not more than the variance of the degree sequence + maximum frequency of the degree sequence. (-1, 4.)*

Disproved by **William Staton, OleMiss** *6. 88.*

12. *radius is not more than 1 + Randic index. (3.)*

Gilles Caporossi and Pierre Hansen *conjecture on the basis of performance of AutoGraphiX that for graphs which are not even paths the Randic index is more than the radius and they prove their conjecture for trees. 6. 97.*

13. *radius is not more than the average distance + sum of reciprocals of degrees. (3.)*

Disproved by **James B. Shearer, IBM, Yorktown Heights***, 10. 89.*

14. *radius is not more than average distance + Randic Index. (3.)*

15. *radius is not more than the variance of the degree sequence + Randic Index. (3, 4.)*

Gilles Caporossi *and* **Pierre Hansen, University of Montreal** *noticed that even paths with* ≥ 22 *vertices are counterexamples to this conjecture and for* $n \geq 26$ *even paths are counterexamples to the next conjecture. 3. 97.*

This conjecture was refuted with aid of AutoGraphiX, comp conj. 3. 6. 97.

They think that these graphs may be the only counterexamples to either of these two conjectures. If that is the case then from Graffiti's "point of view" these conjectures would be correct. One its principles is that given a choice one should rather make a false conjecture than to risk missing a good one. Sometimes when I referred to missing counterexamples because of numerical errors that could be the real reason. I mention this now, because further Pierre wrote:

"There are many interesting questions and insights into the foundational questions of why and when a mathematical result is interesting.

For the fun, let me mention two connections with other work. First, Karl Popper discusses in "The Logic of Scientific Discovery" the selection of conjectures in empirical science. He suggests the best are the simplest ones, which exclude the largest number of possible facts (i.e. which have the most potential falsifiers). There are many developments in that book and later works of him and his school. There might be some connections with the idea of finding the strongest version of the conjecture as explained in the first pages of "Written on the Wall". Then the idea of judging the interest of a conjecture by the distance between concepts it involves is reminiscent of Andre Breton's, theory of the strength of a poetic image, in the "First Manifesto of Surrealism". He considers the image the more potent, the more it brings together different contexts and cites Lautriamont on "the encounter on an operating table of a sewing machine and an umbrella". "

Incidentely about three weeks ago Micha Hofri (see 783) wrote to me that Graffiti shows that Picasso was wrong on one occasion. Picasso, who either did not share Breton's enthusiasm for automation or (what seems to me more likely was trying to set higher standards,) said that: "computers are useless; they can only give answers." I was familiar with this quote, and I am too fond of quoting it in the same context.

Popper, as far as I know was the only scientist who unbashfully gave this issue the attention it deserves, and his Falsifiability Criterion may be the first epistemological principle discovered since Ockham Razor. I personally like more, his "Conjectures and Refutations." But the obvious relevance of surrealist ideas somehow escaped me. Breton after all referred to automatism as eagerly as to Freud, and *after he quit his study of medicine he started his apprenticeship in arts from joining the Dadaists who were probably the first to experiment with automated writing. Putnam, who wrote a lot about models of mind, classifies machine intelligence attempts as engineering tasks, and certainly there is a point to it, but I personally see it more as a process closer to simply, writing.*

Not that artists can always outdo mathematicians. When I asked John Conway what makes a good conjecture, he said without any hesitation: "it should be outrageous." I asked the same question many others but Conway was one of but two persons who did answer it. The other was Peter Frankl who said that it is the history of a conjecture which matters. As conspicuous it was, I gave it a lot of thought and the idea was incorporated in the Dalmatian version of Graffiti described in my "accelerator" paper which I plan dutifully finish any year now.

Paul Erdos' answer was his usual "let's leave it to Radamantes," which can be interpreted the same as Peter's answer, but I think that he simply tried to dismissed the question as nonsense, to my disappointment. Of course soon after I started to ask my question it became quite clear to me that the knowledge of aerodynamics may be quite irrelevant to the flying abilities. But of course it is necessary for programs to take-off.

Rene Thom, asked the same question, said that many oustanding mathematicians work on very difficult, but not very significant problems. 3. 97.

Who wrote this cute poem, about the lizard which can run on water. The poems ends with:

"but if he only stopped to think, oh, how he does it,

he would sink."

Two days after I wrote the above, the "Scientific American" published a short article of two Harward engineers who explaind the mystery. They videotaped several Costa Rican basilisks running on water and even constructed mechanical models to test their theory. Whatever is the theory, it helps if you don't weight more than a few grams, (but that's one up for engineers.)

Making of conjectures however is not so easy to videotape, particularly that truly good ones are much more exotic than basilisks. Reading the article I recalled that once, Professor Erdos did (after all, but only in a sense) answer my question, by saying that "one has to work on a conjecture." He did not mean a solution because he quoted on the occasion Gauss's elaborate computation of frequency of primes on the basis of which he conjectured the Prime Number Theorem.

On the other hand, Hardy kept repeating the question about a formula for $\pi(n)$ *and Professor Erdos lost a* 10 *dollar bet to me to me by claiming that Hardy could not ask such an absurd question. That's one up for John Conway. I have the bill signed in Erdos's unmistakable handwriting and I will throw it in for a simple solution to his 100 dollar question from conjecture 63. 8. 97.*

"Mathematician's Apology" and Hilbert's Paris lecture are about the only two written mathematical sources (which I know of), addressing the issue of significance of conjectures. 10. 97.

16. *radius is not more than the average temperature + Randic index. (-1, 3.)*

Disproved by **Gilles Caporossi** *and* **Pierre Hansen, University of Montreal**. *See conj. 3 and 15.*

17. *radius is not more than the variance of the degree sequence + maximal frequency of the degree sequence. (4.)*

Disproved by **James B. Shearer, IBM, Yorktown Heights,** *10. 89.*

18. *radius is not more than the number of vertices of maximum degree + the maximum frequency of the degree sequence.*

Disproved by **Shui-Tain Chen, University of Houston,** *March 88.*

February 19. 87.

19. *Let a be the smallest eigenvalue of a graph G. Then - a is* $\leq$ *Randic index of G. (3.)*

This conjecture was proved by **Favaron, Maheo and Sacle, University of Paris-Sud.**

comp also conj's 20, 21, 27 and 28.

[FMS2] O. Favaron, M. Maheo and J-P. Sacle, Some Results on Conjectures of Graffiti - Part II.

20. *The number of positive eigenvalues of a graph is not more than their sum. The sum of absolute values of eigenvalues of an integer valued matrix is greater or equal to its rank i.e the number of nonzero eigenvalues, [F2]. This is also a partial solution to conjecture 21.*

[F2] S. Fajtlowicz, On Conjectures of Graffiti, II. Congressus Numerantium 60 (87), p. 189-197.

21. *The number of negative eigenvalues of a graph is not more than the sum of its positive eigenvalues.*

comp. conj 20.

22. *Let e be the largest negative eigenvalue of a graph. Then -e is smaller or equal to the independence number.*

The Paley graph with 101 vertices is a counter-example. **Noga Alon, Tel Aviv University and Bellcore.** *June 88.*

Alon and myself noticed that the conjecture is true for triangle-free graphs. Indeed for these graphs the independence number $\geq$ maximum degree $\geq$ largest eigenvalue $\geq$ absolute value of any eigenvalue. It seems of interest to characterize all exceptions to this conjecture. Noga Alon suggests that they should have few distinct eigenvalues. 6. 88.

Is 22 true for K_4-free graphs? 2. 96.

23. *rank of the adjacency matrix is not more than the rank of the distance matrix.*

Even cycles C_n, $n \geq 6$ or more are counterexamples. **James B. Shearer, Thomas J. Watson Research Center, Yorktown Heights, May 88.**

The counterexample is C_6 (and all bigger even cycles) but in spite of their simplicity, I think that counterexamples are rather rare graphs and they must have a highly regular structure. Perhaps it will be even possible to describe all other counter-examples if there are any.

James B. Shearer noticed that k-dimensional cubes Q_k are also counterexamples to the conjecture. For large k, Q_k are also counterexamples to 26, 29 and 33. Shearer also remarks that by attaching various small graphs to large even cycles we can obtain, by using the interlacing theorem, examples which are not highly regular. If the small graph is an edge then the resulting graph is a counter-ample to 34. July 88.

Perhaps one could prove therefore that counterexamples must contain a large subgraph with a nice structure. For trees the distance rank is n, [GP], hence a counterexample must contain an induced cycle. For example we could ask: suppose that the size of the largest induced cycle is 4. Does G contains a large cube?

Noga Alon, Michael Saks, Paul D.Seymour, Bell Communication Research, and Peter Winkler, Emory University also noticed that for $n \geq 3$, Q_n are counterexamples to this and conjectures 29 and 33; for $n \geq 5$ they are counterexamples to 26. July 88. Shearer has found further rather common counterexamples which can be obtained from cubes by deletion of a relatively few vertices. In particular one can obtain this way examples in which the largest block is C4. August 88. I decided that I will be better off leaving conjectures to Graffiti, see conjectures 303-308.

I was just rereading Shearer's letters from this period who on one occasion found that C_5 was a counterexample to a couple of conjectures with numbers as high 350 or so. As rule I was adding new graphs to the database of the first version of program only (though not always, since sometimes examples were very complicated or very large) if they were counterexamples to previous conjectures. 5.95.

24. *average temperature is not more than the number of negative eigenvalues.*

This and the next conjecture were proved by **Shearer.**

25. *average temperature is not more than the number of negative eigenvalues of the distance matrix. (-1.)*

see 24.

26. *Sum of reciprocals of degrees is not more than the rank of the distance matrix. (1.)*

This was disproved by **Alon, Saks, Seymour, Shearer and Winkler.**

27. *The standard deviation of the degree sequence $\leq$ Randic. (3,4.)*

James Shearer, IBM Research Center in Yorktown Heights, *proved that if G has no isolated vertices then the Randic index of G is greater or equal to $\frac{\sqrt{n}}{2}$. It is the first proof that Randic index goes to infinity together with n. The* *bound is not far off from the best possible since in stars it is equal to* $\sqrt{n}$. *February 88.*

Noga Alon, Tel-Aviv University, *proved that if G has no isolated vertices then its Randic index is greater or equal to* $\sqrt{n} - 8$. *July 88.*

Odile Favaron, Maryvonne Maheo and Jean-Paul Sacle, University of Paris-Sud *deduced from their solution of conjecture 19, a positive answer to this conjecture assuming that G is triangle-free. January 89.*

Bela Bollobas, Cambridge University and L. S. U, and Paul Erdos, everywhere, *proved that if G has no isolated vertices than the Randic index of G is greater or equal to the Randic index of a star with that many vertices. They ask what is the minimum value of the Randic index for graphs with a given minimum degree. November 89.*

Bela Bollobas and Paul Erdos *proved that if G has m edges then the Randic index* $\geq 1/4(8m+1)^{1/2}$, *[BE]. The paper contains several other bounds for the Randic index and their generalization to hypergraphs. May 93.*

comp 3, 19, and 28.

[BE] Bela Bollobas and Paul Erdos, Graphs of Extremal Weights, Ars Combinatoria 50 (1998) 225 -233.

28. *Randic index is not more than the sum of positive eigenvalues.*

This conjecture is false, but working on a counterexample I formed a conjecture that for trees the value of Randic index is close, and seem to be correlated with the sum of the positive eigenvalues. Since eigenvalues of the adjacency matrix of graphs of hydrocarbons are approximation of the solutions of their Schrodinger equation, I would conjecture that the boiling point of single-bonded, acyclic hydrocarbons depends only on the sum of positive solutions of the Schrodinger equation. I discussed my *question with Professor* **Matcha from UH Chemistry Department**, *who told me that it is very sensible and plausible for this class hydrocarbons, while certainly not true for arbitrary molecules. November 89. comp 845.*

Comments following conj 20, show that the conjecture is true for graphs of rank n, since Randic index is always smaller or equal to n/2 and the sum of positive eigenvalues is equal to half of sum of absolute values of all eigenvalues. It is interesting that the occurance of zero in the spectrum of a graph corresponding to alternating (i.e., bipartite) hydrocarbons. indicates a chemical instability, [CDS], p 232.

comp 3, 19, 20, 21, and 27.

29. *The Randic index is not more than the number of negative distance eigenvalues. Disproved by* **Alon, Saks, Seymour, Shearer and Winkler**. *July 88.*

30. *The number of positive distance eigenvalues is not more than sum of temperatures of vertices. A counterexample is D6(B6).* **James. B. Shearer,** *July 88.*

31. *The negative of the largest negative distance eigenvalue is not more than the independence number. A counterexample is D2(B4).* **James B. Shearer**, *7.88.*

32. *The negative of the largest negative distance eigenvalue is not more than the matching number.*

33. *The negative of the largest negative distance eigenvalue is not more than the chromatic number. Disproved by* **Alon, Saks, Seymour, Shearer and Winkler.** *comp 23.*

34. *n - rank of the distance matrix is not more than the maximum frequency of the distance. Disproved by* **James B. Shearer**, *comp 23.*

35. *The diameter of a graph is not more than the negative of the largest negative distance eigenvalue.*

This and the next conjecture follows from interlacing theorem. **James B. Shearer**, *7.88*

36. *The diameter of a graph is not more than the number of negative eigenvalues of the distance matrix.* **James B. Shearer**, *see 35.*

37. *Radius is not more than the sum of positive eigenvalues. Proved in [FA2]. Also partially proved by* **Zang Shu, Beijing Institute of Technology.**

38. *The variance of the distance matrix is not more than the negative of the smallest eigenvalue.*

39. *The deviation of the distance matrix is not more than the number of positive eigenvalues.*

40. *The deviation of the distance matrix is not more than the number of negative eigenvalues.*

41. *chromatic number + radius is not more than the maximum degree + the frequency of maximum degree. Disproved by* **Shui-Tain Chen, UH** *11. 87.*

42. *chromatic number + the average distance is not more than the maximum degree +the frequency of maximum degree. Disproved by* **Shui-Tain Chen, UH**, *11. 87.*

Conjectures for regular graphs (43:62)

43. *Let s be the smallest eigenvalue of G, and m its matching number. Then - s is smaller or equal to m.*

Proved by **Favaron, Maheo and Sacle, University of Paris-Sud,** *[FMS2].*

44. *The second largest eigenvalue is not more than the independence number.* **Noga Alon** *found counterexamples of arbitrary high connectivity. Not connected counterexamples are easy, because the second eigenvalue may be equal to the largest one.*

45. *The second largest eigenvalue is not more than the matching number. Proved by* **Noga Alon**, *5.88.*

46. *The smallest positive eigenvalue* $\leq$ *number of vertices of the center of a graph. Disproved by* **James B. Shearer.** *May 88. See also 58.*

47. *The smallest positive eigenvalue* $\leq$ *number of vertices on the boundary. Disproved by* **Peter Puget, The University of Puget Sound**. *November 88. The counter-example has 18 vertices but it is rather complicated.*

48. *The sum of positive eigenvalues* $\leq$ *largest eigenvalue of the distance matrix.* **Peter Puget, The University of Puget Sound.** *September 88.*

49. *-largest negative eigenvalue* $\leq$ *minimal frequency of the distance matrix.*

50. *The number of zero eigenvalues* $\leq$ *smallest mode of the distance matrix.* **s.f.** *April 87.*

51. *The number of zero eigenvalues* $\leq$ *the number of vertices in the center of the graph.* **s.f.** *April 87.*

52. *The number of zero eigenvalues* $\leq$ *the number of vertices in the boundary of the graph.* **s.f.**

53. *average temperature* $\leq$ *minimal frequency of the distance matrix. Proved by* **Shui-Tain Chen, U. of Houston**. *April 87.*

54. *If* G *is a regular graph of degree* d *with* n *vertices then* n/d *is not more than the rank of* G*. Proved independently by* **Noga Alon** *and myself, [F4]. 5.88.*

[F4] S.Fajtlowicz, On conjectures of Graffiti, Part 4, Congressus Numerantium.

55. *The second largest eigenvalue is not more than the minimode of distance. Disproved by* **Peter Puget** *11. 88, and later independently by* **Favaron, Maheo and Sacle***. 1.91*

56. *The second largest eigenvalue is not more than the number of centers. Disproved by* **Halina Bielak, U. of Marie Curie-Skladowska, Lublin, Poland***.*

It seems that I lost 57. Perhaps someone has an original version.

58. - *largest negative eigenvalue is not more than the number of centers.*

Disproved by **Peter Puget**

59. *Diameter is not more than the sum of positive eigenvalues. Disproved by* **Peter Puget**

60. *mode of distance is not more than the independence number. Disproved by* **Alon, Saks, Seymour and Winkler***.*

61. *mode of distance is not more than the matching number. Disproved by* **Alon, Saks, Seymour and Winkler***,who also proved that the conjecture is true for regular graphs of high degree, $\geq$ 10 is enough.*

62. *If G is a regular graph of degree d then the average distance is not more than n/d,*

comp 127.

Peter Puget *proved the conjecture for graphs of diameter ≤ 3 and constructed an infinite family of graphs of diameter 4 and even F such that average distance = 2+ (3d+4)/(4d^2+10d+6) and n/d = 2 + (3/d). January 89.*

Puget thinks that if there is a counterexample then d is close to n/2, even though the difference in this case will not be too large. I thought that the best bet is d which is rather small compared to n. Here is my reason: Let $H_n(d_1, .., d_n)$ be a sequence of trees defined recursively as follows: H_{n+1} is constructed by taking *two copies of H_n and joining their centers by the path of length d_{n+1}. If we can assure that the average distance is sufficiently large then the resulting tree could be "regularized" though d can not be too small, because for $d = 3$ the maximal average distance in a connected graph is (n+1)/3. I think that this problem is of interest for its own sake, because H-trees are used in layout problems for VLSI's, [UL]. I do not know though, how to select d_i's. Perhaps it is actually impossible. In this case I would believe that the conjecture is true and that it can be proven using this result by proving additionally that every regular graph G contains a tree with the average distance well approximating the average distance of G. February 89.*

Let T be a spanning tree of G minimizing the difference between the average distances of T and G. What is the maximum of this difference in the class of all regular graphs of a given degree? 2. 96.

Tamara McColgen, University of Mississippi, *proved the conjecture for Hamiltonian graphs. July 89.*

Peter Puget, *[BRS] proved the conjecture for super regular graphs, i.e, such that that for every vertex v and integer k the number of vertices at distance k from v does not depend on the choice of v and k. The paper also describe the methods (involving computers, catalogs of graphs and the program Reduce) used to disprove several conjectures of Graffiti about eigenvalues of regular graphs . The paper also contains information on residue in regular graphs, needed to disprove the related (in view of 2, 69 and independence $\geq n/3$) conjecture 93 , interesting examples and near counterexamples, and finally some related conjectures.11, 89.*

The conjecture was proved by **Peter Puget,** *see 127. 4. 98.*

[BRS] Robert A. Beezer, John Riegsecker and Bryan A. Smith, On conjectures of Graffiti concerning regular graphs, Technical Report N0 89-1, August 89, The

University of Puget Sound.

63. *The average degree of a triangle-free graph is not more than its Randic Index. Replacing in the definition of Randic the definition of the weight of an edge by 2/(p+q) we get an invariant which will be called here* ***harmonic****. Many (though not all, see for example related 27) conjectures involving Randic on the right side of the inequality remain true (as conjectures) if Randic is replaced by harmonic and as a matter fact* **Shearer** *proved that if graph is triangle-free then the average degree is not more than the harmonic.*

Turan's Theorem for triangle-free graphs is equivalent to the statement that the average degree is not more than n/2. It is easy to see that harmonic $\leq$ Randic $\leq$ n/2. Thus Shearer's bound is never worse than Turan's. Is there a better triangle-detecting test which can be performed in a time ctimes n^2 ? It is known that if the largest eigenvalue = L $\geq$ n/2 then the graph must contain a triangle. This sometimes produces a better test but I do not know if it can be performed in the time cn^2; note that to perform this test we do not have to compute the exact value of L. Similar questions can be asked for K_4-free graphs.

$deg(v_i)+deg(v_j) \geq n$, for every pair of adjacent vertices is a better test than all of the above. **James B. Shearer**. *February 88.*

64. *If G is triangle-free then the mode of Degree $\leq$ then the Randic Index. Disproved by* **James B. Shearer**. *July 87.*

65. *If G is triangle-free then the mode of Degree $\leq$ the matching number. Disproved by* **Thomas Spencer, Rensselaer Polytechnic Institute** *February 87.*

66. *If G is triangle-free then radius $\leq$ the sum of reciprocal of degrees.*

Disproved by **Thomas Spencer, Rensselaer Polytechnic Institute** *February 87. I wrote in [FA2] that it was the first counterexample with over 100 vertices but that was incorrect. The first large counterexamples were found in [EPS]. Anyway Thomas Spencer found later a counterexample with 20 vertices. [EPS] dealt with the conjecture average distance $\leq$ sum of reciprocal of degrees, and essentially the problem of asymptotic bounds is settled there.*

67. *chromatic number of a triangle-free graph is not more than the maximum frequency of its degree sequence (mfd.).*

Disproved by **William Staton, U. of Mississippi**, *July 87.*

Staton skillfully modified Mycielski's construction of triangle-free graphs with high chromatic number to get a counterexample of chromatic number 5. We would like to know if the conjecture is true for graphs of chromatic number 3. This imposes very strong conditions on the degree sequence of a potential counterexample. Comp. 119.

Staton *proved the conjecture for chromatic number 3. November 88.*

Every triangle-free graph can be embedded into a triangle-free graph with the maximal frequency of degree $k = 3$. This settles the conjecture for every k, but graphs obtained by our method are very large. What is the smallest integer $c = c(n,k)$ such that every triangle-free graph with n vertices and maximum frequency of degree k has chromatic number $\leq c$? A question closely related to our method, see [EFS] is what is the upper bound for $u(n,k)$ = minimum degree in n-vertex graphs with the mfd = k? Paul Erdos and S.F. November 88.

Paul Erdos *and* **William Staton** *constructed a counterexample with 19 vertices based on Grotzsch graph. It is not obvious how to extend it to counterexamples* *with a higher girth, comp 119. We also do not know whether it is the smallest counterexample to this conjecture . November 88.*

Professor Erdos asks what is the best lower bound for the independence in K_p-free graphs with $mfd \leq k$. February 89.

If p=3 then the independence is at least $\geq n/k$. It is one more reason to ask for a bound for the minimum degree.

Paul Erdos *asks what are the best lower bounds for the independence number in graphs containing no k-element cliques nor more than l vertices of the same degree. 2.89.*

Conjecture 67 motivated Erdos and myself to define the following numbers: N(k) is the smallest integer N such that every graph with N vertices contains an induced regular subgraph of order k,

and

n(k) is the smallest integer n such that every graph with n vertices contains an induced regular subgraph of order k or more.

The numbers are obviously not greater then the diagonal Ramsey numbers, but perhaps as hard or even harder to evaluate. They numbers seem to me of interest even for very restricted classes of graphs like Cayley, Paley, or even bipartite. [FMRS] contains proofs that

$n(4) = 7$.

$N(4) = 8$.

n(5) is not less than 12.

N(5) is not less than 19.

N(6) is not less than 18,

and

if p is prime than N(p) is not less than (p-1)(p-1) + 1 **William Staton 3. 91**.

$N(5) \geq 20$. A 19-vertex example is the Cayley graph of cyclic group with the generating set 1,2,3,16,17,18. s.f 9.93.

Brendan McKay, Australian National U., Canberra *found many 20 vertex examples, but none can be extended to 21 vertices. He thinks that N(5) is 21 or 22. Nov. 93.*

N(5) is at most 31. McKay established the upper bound by using a program to generate all largest graphs containing no K_4, no induced C_5 nor 5 element independent sets. They have 15 vertices and there are two of them. Nov.93.

'94. The first two formulas were proved independently **Paul Erdos** *and* **Yoshiharu Kohayakawa, IME, Cidade Universitária, São Paolo, Brasil** *'94.*

1.97. **Brendan McKay** *verified with a computer that n(5) = 17, and that there are 954 non-isomorphic, 16-vertex critical graphs. Perhaps any proof of n(5) = 17 must be quite long. The most interesting problem of this kind is Haken's conjecture that every proof of 4-color theorem must have length approximately the same as the one found by him and Appel. 1. 97.*

What is the smallest number n = rp(k,l) such that every graph with mfd $\leq$ p contains a k-element clique or an l-element independent set? **W. Staton and S. F.** *March 89.*

I noticed that for all random graphs $G_{n,p}$ in Graffiti the value of the Randic index is very close to n/2. These values seem to me much closer (at the first thought) than I would expect from the distribution of the degree sequence in these graphs, (it is easy to show that value of the Randic index = n/2 iff every component is regular and G has no isolated vertices.) Let c be constant and let G be a graph *in which which (n/2 - Randic index $\leq$ cn. What is the order of the largest induced regular subgraph of G and what is the order of the largest induced subgraph of G whose components are regular? 10. 89.*

3. 97. A proof of $R \leq n/2$ where R is the Randic Index: let us duplicate each edge by orienting it in two possible ways. Then $2R$ is the the sum over oriented edges of $1/\sqrt{(pq)}$, where p is the degree of the head and q the degree of the tail of an edge. By Cauchy inequality the sum is not more than the sum (over the same set) of reciprocals of degrees. Since the summond $1/p$ occurs p times, the sum is at most n. 3. 97.

Roger Entringer, Laszlo Szekely, The University of New Mexico *and* **Paul Erdos** *constructed graphs of order n containing no induced regular subgraphs of order n^{1-c} and they ask if c can be as large as 1/2 and they offer the following conjecture: Every graph G of order n contains an independent or a complete subgraph of order 100logn or a complete or independent subgraph of order clogn whose all vertices have the same degrees as in G. Nov.89.*

11. 96. [FCH] contains information that Bollobas (unpublished) constructed such graphs. I also found there a 100 dollar problem of Erdos and McKay related to problems in this section: Let f(n,c) denotes the largest integer m such that a graph G on n vertices containing no cliques nor independent sets of size clogn must contain an induced subgraph with exactly i edges for each i, $0 < i \leq m$. Prove or disprove that $f(n,c) \geq cn^2$. 11. 96.

Paul Erdos *asks, if there is a constant c such every graph with n vertices containing no cliques nor independent sets of order 10logn contains a regular subgraph with c*logn vertices. 2.91.*

It is not obvious what lower bounds for numbers n and N can be obtained from random graphs, but **Noga Alon** *told me that he proved that these bounds are better by at least a constant factor than the corresponding bounds for Ramsey numbers. 12.92.*

Let C be a coloration of all k-element subsets of a set X with c colors. A subset Y of X will be called regular if for every color r every element of Y belongs to the same number of subsets of color r. Let $\{p,k,c\}$ *be the smallest integer n such that for every c-coloration of all k-element subsets of the set* $X=\{p,p+1,\ldots n\}$ *there is a regular subset Y of X such that the number of elements of Y is not less than min(Y).*

The existence of numbers $\{p,k,c\}$ *follows from the existence of Paris-Harrington numbers - the first interesting combinatorial result unprovable in Peano arithmetic.*

I think that the numbers $\{p,k,c\}$ *are much smaller but perhaps their existence is also unprovable in Peano arithmetic.*

A hypergraph is a system (X,F) where F is a family of subsets of X. If all sets in F have the same cardinality then (X,F) is called uniform. If F contains all sets of a given cardinality then it is called complete.

Let H be a class of hypergraphs containing all complete hypergraphs. Let n= H(p,k,c) be the smallest integer such that for every c-coloration of of all k-element subsets of X=p,p+1,..n there is a subset Y of X such that cardinality of Y is not less than min(Y) and for every color the hypergraph induced on Y by subsets of this color belongs to H.

If H is the class of all complete hypergraphs then numbers H(p,k,c) become Paris-Harrington numbers and if H is the class of all regular hypergraphs then they become $\{p,k,c\}$.

By using various H it is probably possible to construct new interesting statements unprovable or having very long proofs in Peano arithmetic. [L] contains a unified approach to unprovable statements and references to previous papers. 12.92.

Paul Erdos offers **100 dollars** prize for a proof or disproof of the conjecture that the numbers n(k) or N(k) grows exponentially i.e., that asymptotically n(k) or $N(k) \geq (1+c)^k$, with c 0. 1. 94.

For a group G let $R(G)$ be the smallest integer n such that every graph with n or more vertices contains an induced subgraph whose automorphism group contains G. The simplest interesting question seem to be what is $R(S_5)$. $R(S_4)$ is at most 18 and probably it is 18, because Ramsey number R(4,4) is 18 and the 17-vertex Paley graph is the unique critical graph for $R(4,4)$.

The numbers $R(G)$ are related to the numbers n(k), at least for small k. For example n(4) =8 implies that $R(Z_2^2) \leq 8$.

If p is a prime then $R(Z_p) \geq p$, because the orbit of any vertex must have exactly one or p elements. $R(Z_3) = 6$, by R(3,3) =6, the critical graph being C_5.

If p is a prime and v is vertex such v is not a fixed point of an automorphism of order p, then either the graph or its complement contains a p-cycle C. Two vertices of C are adjacent iff every two vertices of the same distance in C are adjacent. Thus for a prime p, $R(Z_p)$ is the smallest integer n such every graph on n or more vertices contains an induced Cayley graph of Z_p, and in particular $R(Z_5) = N(5)$

$R(Z_4) \leq 8$ by $N(4) = 8$ and it seems that is exactly 8 because of two unique critical graphs described in [FMRS], but I did not check it carefully. One of these graphs W is the wheel of C_5 plus an isolated vertex and the other is its complement (in the year of Hale-Bopp comet it is actually nicer to draw the complement.) If

G is the direct product of Z_2 with itself then $R(G) \leq 7$ because both W and its complement contain $K_4 -$ edge. 1. 97.

[FCH] F. R. K. Chung, Open problems of Paul Erdos in Graph theory, preprint 96.

[L] M. Loebl, Unprovable combinatorial statements, Discrete Math, 108(92), 333-342.

[EFS] P. Erdos, S. Fajtlowicz and W. Staton, Degree sequences in the triangle-free graphs, Discrete Mathematics, 92 (91), 85 - 88.

[EFRS] Erdos, P., Faudree, R., T. J. Reid, R. Schelp, and W. Staton,"Degree sequence and independence in K(4)-free graphs", Discrete Mathematics 141, 285-290, 1995.

[FMRS] S. Fajtlowicz, T. McColgan, T. Reid and W. Staton, Ramsey Numbers for Induced Regular Subgraphs, Ars Combinatoria, 39,149-154, 1995.

68. *If G is triangle-free then the matching number $\leq$ then the maximal frequency of the degree sequence. Disproved by* **Thomas Spencer, Rensselaer Polytechnic Institute** *February 87.*

November 26. 87.

69. *Residue is not more than the independence number.*

This is really a joint conjecture of Graffiti and myself, because I began to anticipate this conjecture writing the code for residue. 11. 87.

Residue is defined as follows: Let D be the degree sequence D of a graph in non-increasing order. Let d be the first term of the sequence. Then the derived sequence is obtained from D by deleting the first term and subtracting 1 from d following d terms of D. Havel and Hakimi proved independently that the derived sequence is also a degree sequence of a graph. Hence if the operation of taking the derived sequence *is repeated (sorting new sequences each time) then process terminates and we end up with a sequence of zeros. The number of these zeros is called the residue of the original graph.*

Favaron, Maheo and Sacle, *proved this conjecture and hence also 98 and 80. Their proof is more than 10 pages long and is rather complicated. They also proved that some previously known lower bounds for the independence are also valid for the residue. One of them is usually called the Turan bound - sum 1/(d_i +1) where d_i is the degree of the ith vertex. This indeed is a consequence of Turan Theorem or essentially Erdos's Majorization Theorem, though I don't think this was known before the proof was published independently by Caro, in '79 [YC], and then by Wei in 81. June 88.*

Shearer *conjectures that every performance of Maxine (see [GJ] and [GR]) will find an independent set whose size is at least as large as the residue. July 88.*

Jerrold Griggs, University of South Carolina *proved the conjecture of Shearer. February 89.*

Jerrold Griggs and Daniel Kleitman, MIT *found a shorter proof of 69. Their proof is still quite technical, and it is based on the same main lemma as the proof from [FMS]:*

(*) If D and E are realizable as degree sequences and D dominates E then residue of D $\geq$ the residue of E. D ***dominates*** *E means that their sums are the same and the sorted partial sums of D are greater or equal to the corresponding sums of E. September 90.*

Pierre Hansen and Maolin Zheng *told me that they also found a new proof of 69 (in February 90.) It is though longer than the proof of Griggs and Kleitman. September 91.*

Let D be the degree sequence of a graph, and let b(D) = min{a(G): D(G) = D}, where a(G) is the independence number of G. One can think of b(D) as the best lower bound for the independence number, which can be derived from the degree sequence. This invariant was de facto proposed in [FMS] who first asked if b(D) is equal to the residue, but later found counterexamples to their question.

Let D be the sequence derived from G by Havel-Hakimi reduction. Is it true that b(D*) is smaller or equal to b(G)? If the answer is yes, it easily implies that the residue is not more than the independence number. Indeed*

$r(D) = r(D^*) \leq b(D^*) \leq b(D) \leq a(G)$ *for every G with D(G) = D.*

Another question concerning b(D) is how to to compute it, and in particular if this problem in NP-hard? 4. 95.

I think that lemma (*) is also true for the function b(D).

Since the independence number of the line-graph of G is the matching number of G, one can ask very similar questions replacing everywhere the independence number by the matching number. Residue is often quite a good bound for the independence, compare for example 448. Perhaps it is equally good as a bound for matching. As far as the distribution of primes is concerned perhaps it is even more interesting what is b(PR[2..n]). **Erdos and Staton** *proved in 448 that it is at least cn/logn, where c a constant close to 2/3. s.f 4.95.*

Concerning the simpler proof of 69, perhaps this can be done like this: Havel-Hakimi Theorem can be proved by repeated switching of certain pairs of edges to make sure that a vertex of highest degree is adjacent to the next highest. To prove the conjecture it is enough to perform this operation without increasing the independence number. Perhaps this idea is present in both proofs, because switching seem to correspond to the covering relation in the dominance order of degree sequences.

Below is a fragment from a letter of Shearer which contains a simple, but very innovative application of the probabilistic method.

"Yet another way of proving this result (i.e, Turan bound, s.f) is to apply the random greedy algorithm i.e, order the vertices of G at random then consider each in turn placing it in the independent set if it is unconnected to the points previously placed. Clearly a point will end up in the independent set if it precedes its neighbors in the random order i.e, with probability greater or equal to 1/(deg(i) +1) proving the average size of the independent set is at least sum 1/((deg(i) +1)). Maxine appears to be related to conjecture 69. It is easy to show that residue is greater or equal to sum 1/(deg(i)+1) and it is tempting to conjecture that Maxine is greater or equal to residue." [SH1]

In a more recent letter Shearer explains that there are two somewhat different probabilistic proofs and points out that the algorithm corresponding to the first proof may produce an independent set, which is not maximal:

" You can number the vertices from 1 to n at random and include a vertex in I if it has a smaller number than any of its neighbors. Then the probability that a vertex, v, is included in I is exactly 1/(deg(v)+1). Alternately you can choose a random order of the vertices and consider each vertex in turn placing it in I if none of its neighbors have already been placed in I. Then the probability that a vertex, v, will be included in I is at least 1/(deg(v)+1). (Since v is certain to be included if it is first in the random order among v and its neighbors and v may be included otherwise.) The second algorithm is equivalent to the random greedy algorithm (include random vertices in I and throw out their neighbors until all the vertices are gone)." [SH2].

Shearer's proofs and his conjecture proved later by Griggs inspire "converse" questions: Given an algorithm A for computing an independent set, what is the

domain *of A i.e, the class of all graphs A* for which every performance of this algorithm results in a maximum independent set. Let us consider a special type of (exclusion) algorithms, which start with I as the set of all vertices, and then delete some of them according to a certain rule until I is independent. An example is Maxine, and another example of interest because of conjecture 2 is: exclude vertex of minimum eccentricity (or a vertex whose deletion will maximize the average distance, which are probably different things). A precaution has be taken to extend the definition of eccentricity (or average distance) to disconnected graphs, so that the concepts remain meaningful after deletion of a cut-point.*

We can dually define inclusion algorithms by analogy with MIN - the algorithm which starting with the empty set I, adds to I vertices of minimum degree, which are not adjacent to vertices already in I. I incorporated lately into Graffiti both Maxine and MIN as invariants, (or really tryouts) and it seems that usually MIN performs somewhat better than Maxine.

The concept of the domain is related to Invariant Interpolation Problems which are discussed in conj 814.

The domain of the random greedy algorithm (see the second Shearer's letter) consists of all graphs in which every two maximal independent sets have the same number of elements. The domain of Maxine contains all graphs in which the independence number is equal to the residue, but there are other graphs with this property. First examples were found by **DeLaVina***, and other examples are PR[2..n], comp conj. 801.*

I found that in [CST] authors discuss a closely related idea namely combinatorial problems which can be solved by greedy algorithms. Summer 96.

I have no idea, what is the domain of MIN. The metric version of MIN, corresponding to the "delete a vertex to increase the average distance" version of Maxine is: starting with the empty set I, place there the first vertex of maximum eccentricity, which is not adjacent to any of the vertices already in I, etc.. 5.95

Is it known whether MIN produces independent set of the size of residue?

[C] Yair Caro, New results on the independence number, Technical report Tel-Aviv University 1979.

[CST] Y. Caro, A. Sebo and M. Tarsi, Recognizing Greedy Structures, to appear in Journal of Algorithms.

[FMS] O. Favaron, M. Maheo and J-P. Sacle, On the Residue of a Graph, JGT 91, p. 39-64.

[GJ], Michael R. Garey and David S. Johnson, Computers and Intractability.

[GR] J. Griggs, Lower bounds for the independence number in terms of the degree sequence, JCT B (83) 22-39.

[GK] J.R. Griggs and D.J. Kleitman, Independence and the Havel-Hakimi residue, Discrete Math. 127(1994), 209–212.

[SH1], a letter from James B. Shearer, July 24, 88.

[SH2], a letter from James B. Shearer, May 8, '95.

[T] E. Triesch, Journal of Graph Theory '95.

70. *chromatic number $\leq$ maximal frequency of coordinates of a maximal clique. comp. 640.*

William Staton, University of Mississippi, *found an example of a graph with chromatic number 8 , in which the maximal frequency of coordinates of every maximum clique is 7. December 87. So far we do not know any examples in which the difference would be greater than 1.*

Using probabilistic methods **Vojta Rodl, Charles University, Prague and Emory University**, *showed that there are graphs in which the difference between the chromatic number and the maximal frequency of coordinates of every maximal clique is arbitrarily large. February 88.*

William Staton *found explicit examples of graphs in which the difference between the chromatic number and the maximal frequency of coordinates of every maximum clique is arbitrarily large. March 88. Both constructions are based on triangle-free graphs in which the maximal frequency is less than n/2. Counterexamples are complements of these graphs. This suggests the following Ramsey-type problem: What is the smallest number rs(l) (RS(l)) such that every graph with that many vertices either contains a triangle or the maximal frequency of coordinates of a maximal (maximum) independent set is at least k. It is easy to see that rs(l)* $\leq$ *RS(l)* $\leq$ *R(3,l) for the corresponding Ramsey numbers. For l leq 5 we have the equality. More directly related to the conjecture are numbers g(l) (G(l)) defined as the smallest n such that for every graph with n vertices, the maximal frequency of coordinates of a maximal (maximum) clique is at least l. It is easy to show that G(l) = (l-1)2 +1 and in particular, for a fixed k there are at most finitely many k-chromatic counter-examples to the conjecture. g(3) = G(3) = 5, but g(4) = 7. I have examples showing that g(5)* $\geq$ *13 and g(6)* $\geq$ *21. The 12-element g(5) - example is the strong direct product of the cycle* C_6 *and* K_2 *and thus has chromatic number 4. However the strong direct product of* C_5 *and* K_2 *has chromatic number 5 . March 88.*

Fan Chung told me once the following concept and the question. An invariant f is related to g if whenever g goes to infinity then f goes to infinity. What is related to chromatic number? Notice that the two invariants in this conjecture are related.

Paul Erdos *constructed for the first time fairly easy counterexamples in which the difference between the two invariants is arbitrarily large. They are based on triangle-free graphs with high chromatic number. This is however for the interpretation of the try-out in question as the maximum, not the maximal clique. February 89.*

Shui-Tain Chen *proved that* $g(5) \geq 15$. *March 89.*

Shui-Tain Chen *proved that* $g(5) \leq 16$. *October 90, [CH2].*

I believe that her proof can be generalized to $g(k) \leq k^2$.

71. *number of cut-edges* $\leq$ *sum of reciprocals of temperatures.*

Proved independently by **James B. Shearer** *and* **William Staton.** *July 88.*

72. *average temperature* $\leq$ *depth and the equality holds true iff the graph is a join of a discrete and complete graphs. February 88. Proved independently by* **Odile Favaron, Maryvonne Maheo and Jean-Francois Sacle.** *June 88.*

73. *The maximum of coordinates of the set of cut-vertices* $\leq$ *the independence number.* **William Staton.** *December 87.*

74. *The maximum of coordinates of the set of cut-vertices* $\leq$ *the matching number.* **William Staton.** *December 87.*

75. *The variance of coordinates of the set of cut-vertices* $\leq$ *the independence number.* **William Staton.** *February 88.*

76. *The mode of coordinates of the set of cut-vertices* $\leq$ *the matching number.* **William Staton and s.f.** *February 88.*

77. *The mode of coordinates of a maximal independent set* $\leq$ *n - residue.* **William Staton and Siemion Fajtlowicz.** *February 88.*

78. *The mode of coordinates of a maximal independent set* $\leq$ *Randic.* **James B. Shearer**, *August 88.*

79. *mean of coordinates of a maximal independent set* $\leq$ *matching. Proved by* **William Staton** *for all independent sets. The equality holds true iff the graph has no edges, but there are graphs like stars in which the two invariants can be arbitrarily close. January 88.*

80. *mean of coordinates of a maximal independent set* $\leq$ n *-residue.*

This follows from 79 and 98, and it was also proved without use of 79 in [FMS]. The equality holds true iff G *has no edges.*

81. *variance of coordinates of a maximal independent set* $\leq$ *maximal frequency of Even. Disproved by* **William Staton.** *March 88.*

82. *range of coordinates of a maximal clique* $\leq$ *maximum of Even.*

The conjecture is valid for all maximal cliques. The equality holds true in cliques but there are also other such graphs. **William Staton***. March 88.*

83. *scope of coordinates of a maximal clique* $\leq$ *maximum of Even.*

William Staton *found a counterexample to the strongest version of this conjecture. However if the clique in question dominates the graph then 83 is true with respect to this clique. March 88.*

84. *The variance of coordinates of a maximal clique* $\leq$ n *- residue.* **William Staton** *March 88.*

85. *The variance of coordinates of a maximal clique* $\leq$ *rank.* **William Staton.** *March 88.*

86. *variance of coordinates of a maximal clique maximum of Even.* **William Staton.** *March 88.*

87. *The variance of the distance matrix* $\leq$ *sum of reciprocals of temperatures.*

88. *mode of the distance matrix* $\leq$ *inverse temperature.* **James B. Shearer***, August 88.*

89. *The average distance $\leq$ inverse temperature.* **James B. Shearer**. *May 88.*

90. *average distance $\leq$ maximal frequency of coordinates of a maximal independent set.*

From Chung's Theorem (see conjecture 2) it follows that the conjecture is true for every maximum independent set; independence equals to the frequency of 0. It still would be interesting to know if 90 is true for all maximal independent sets. Stars show that the average distance is not necessarily less than the size of every maximal independent set.

91. *The variance of the distance matrix $\leq$ the matching number. bf James B. Shearer. February 88.*

92. *The mode of the distance matrix $\leq$ the sum of reciprocals of coordinates of a maximal independent set.*

William Staton *noticed that odd cycles with more than 10 vertices are counterexamples to the strongest interpretation of the conjecture. In his interpretation however both the mode and the independent set must be maximum. June 88.*

93. *The average distance $\leq$ residue.* **Peter Puget.** *April 89.*

94. *The variance of the distance matrix $\leq$ the independence number.* **James B. Shearer.** *February 88.*

95. *The mode of the distance $\leq$ the residue.*

C_9 *is a counterexample with the interpretation of the mode as the largest of modes.* **Odile Favaron , Maryvonne Maheo and Jean-Francois Sacle,** *July 88.*

96. *Let E (D) be the vector whose ith component is the number of vertices at even (odd) distance from the ith vertex.*

Conjecture: The number of distinct components of E is $\leq$ residue. [FMS1]. October 88.

Conjectures for triangle-free graphs (97:104)

97. *The matching number $\leq$ maximal frequency of the vector E from conjecture 96.* **William Staton.** *April 88.*

98. *The matching number is $\leq$ n - the residue.*

This conjecture should be made for all graphs. As I wrote in [FA] I sometimes compute matching by hand and I made a mistake. I am grateful to Favaron , Maheo and Sale who asked me for a counterexample (for all graphs) and drew my attention to the error. From 69 it follows that this conjecture is true for all graphs.

99. *The variance of the distance matrix is $\leq$ n - the residue.* **James B. Shearer.** *February 88.*

100. *Chromatic number is $\leq$ maximal frequency of the vector E from 96.* **Peter Puget.** *June 90.*

101. *The variance of the degree sequence is $\leq$ than the independence number.*

Disproved independently by **James B. Shearer** *and* **William Staton.** *February 88.*

102. *The variance of the degree sequence is $\leq$ than the the mean of the vector E defined in 96.* **James B. Shearer.** *February 88.*

103 *The mean of coordinates of the set of cut-vertices $\leq$ the average distance.* **William Staton.** *February 88.*

104 *The mean of coordinates of the set of cut-vertices $\leq$ the the sum of reciprocals of vector E from 96.* **William Staton.** *March 88.*

105. *If G is a tree then the range of the degree sequence $\leq$ the range of transmission of distance, (i.e the vector of row-sums of the distance matrix.)*

106. *If G is a tree then the average distance* $\leq$ *residue.*

and the equality holds true iff G is a path of order $n = 2 \bmod 3$, **[FMS]**. *April 88.*

107. *A graph G is even-regular if the vector E defined in conjecture 96 is constant. If G is even-regular then the mode of the distance matrix* $\leq$ *radius.*

Vance Faber, Los Alamos National Laboratory *used LANL Cray computer and Reed's program listing all at most 10 vertex graphs to study some of conjectures from this list. Later his students Tony L. Brewster, Los Alamos National Laboratory and Rutgers University and Michael J. Dineen, , Los Alamos National Laboratory and Dept. of Computer Science, University of Victoria, Victoria, B.C used the same devices to systematically test some other conjectures of Graffiti. They tested about 200 conjectures and refuted over 40 of them. Below are numbers of some of the conjectures which passed their test:*

3, 4, 5, 7, 8, 12, 14, 15, 16, 20, 21, 27, 38, 39, 40, 49, 62, 87, 92, 95,105, 117, 118, 123, 126, 127, 128, 129, 131, 134, 136, 142, 143, 150, 152, 154,167,168, 170, 171, 172, 173, 174, 195, 197, 198, 203, 210, 217, 219, 221, 222, 224, 225, 226, 233, 235, 236, 237, 239,275, 282, 283, 284, 287, 290, 291, 292, 295, 300, 303, 308, 312, 313, 322, 345, 346, 350, 351, 354, 355, 356, 362, 364, 366, 367, 368, 376, 378, 400, 402, 404', 416, 418, 422, 424, 425, 433, 540, 543, 545, 547, 548, 549, 552, 553, 554, 555, 558, 565, 567, 568, 569, 573, 577, 582, 584, 585, 589, 592, 597, 603, 611, 614, 616, 629, 632, 633, 662, 665, 670, 680, 692, 698, 699, 700, 712, 714, and 723. August, '90 - August '91. [BDF]

108 *If G is even regular then the average distance is* $\leq$ *the radius.Disproved by* **William Staton. March 88.**

Conjectures for triangle-free graphs (107:116)

109. *Range of the vector E from conjecture 96 is* $\leq$ *residue.*

Peter Puget *found counterexample to this and two other conjectures with the aid of their new program which presents and allows user to modify a graph on a screen. Watching how invariants change prompts a user to get an idea for counterexamples or proofs. 11, 89.*

110. *Even(v) is the number of vertices at even distance from v. Conjecture: If G is triangle-free then range of Even* $\leq$ *Range of Degree. Disproved by* **William Staton.** *March 88.*

111. *If G is triangle-free then* $[n/2] \leq$ *mean of Even.*

Graffiti's original conjecture was that $n/2 \leq$ *mean of Even. The modification is due to* **William Staton** *who observed that if* $n = 3 \bmod 4$ *then the cycles* C_n *are counterexamples. April 88.*

Staton found examples in which the difference is arbitrarily large, but we still have in these graphs that mean of Even $= 3n/7$*; he thinks that perhaps this is the worst case. April 88. Comp.121.*

It is easy to prove that if G is bipartite then the mean of Even $\leq n/2$*. Hence we have another obvious conjecture : If G has chromatic number* $\leq k$ *then the mean of Even* $\leq n/k$*, comp. also 180.*

Since the conjecture is true for trees it is natural to ask what happens when the girth goes to infinity.

If G is a graph obtained from B_{23} *by identifying all points in cosets of the perfect Golay code then the mean of Even/n* $= 254/2048$ *which is about 0.124* **James B. Shearer***. Shearer conjectures that in the worst case the ratio goes to 0. 7. 88.*

Examples similar to Shearer's, but using de Bruijn graphs are used in [CH1] to construct "Moore-like" graphs. Thus again we have an obvious question: what is the minimum of mean of Even for graphs with a given diameter D and a maximum degree k?

Comp. 240 and 245.

Actually, the last question seems to be relevant not only by analogy to these problems. For example if it would be possible to construct Moore graphs of arbitrarily high degree and diameter 3 then this would prove Shearer's conjecture. There are no such graphs, but there exist graphs nearly reaching Moore bound and these should do nearly as well.

Graphs described in comments to conjecture 114 have the above property.

Charles Delorme, L. R. I. University de Paris-Sud and S. F. *12. 89. See also 407.*

112. *Radius $\leq$ maximal frequency of the degree sequence. Disproved by* **Shui-Tain Chen. April 88. James B. Shearer** *found a bipartite counterexample. May 88.* **Shui-Tain Chen** *proved the conjecture for trees, October 88.*

113. *The average distance $\leq$ maximal frequency of the degree sequence.* **Peter Puget, 11. 89.**

114. *Maximal frequency of the distance matrix $\leq$ sum of the components of the vector from 96.*

Graphs described in [DL] are nearly regular graphs of degree d and diameter 3. Since they nearly reach the Moore bounds the ratio of the left side to the right is asymptotically close to d. **Charles Delorme, L. R. I. University de Paris-Sud** *and S. F. 12. 89.*

115. *The number of distinct components of the vector E from 96 is $\leq$ the sum of reciprocals of its components.* **William Staton.** *April 88.*

116. *largest eigenvalue $\leq$ Randic. Proved in* [**FMS2**] *. November 88.*

As it was the case with 63 this conjecture again is a generalization of Turan's theorem in the triangle-free case. Indeed: Randic is always at most $n/2$, and for triangle-free graphs, the size s at most $n/2$(largest eigenvalue). s.f. 10, 89.

Conjectures 117 and 118 are marked on the original list by the letter R. That was reminder to myself that at the time when put them on the list the program could not make these conjectures, because the only counterexamples to stronger refuted conjectures were too large or too complicated, and I could not inform the program about them. From the performance of Graffiti I expected that it would make these conjecture if it knew the counterexamples.

117R. *If G is a connected graph of girth 5 then its average distance is not more than the sum of reciprocals of its degrees.*

This conjecture is a result of one the first conjectures of Graffti. The same thesis, just no assumptions about the girth. The original conjecture was disproved by **Erdos, Spencer and Pach**, *before I started compile this list, but counterexamples were much too large for the program.*

[ESP] Paul Erdos, Joel Spencer and Janos Pach, On the mean distance between points of a graph, Ars Combinatoria, 1988.

118R. *If G is a connected graph of girth 5 then the distance which occurs most often is not more than the sum of reciprocals of its degrees.*

119R. *if G has girth 5 then its chromatic number is not more than the maximum frequency of occurance of a degree (frequency of the mode of the degree sequence).*

Yair Caro, Haifa University, Israel *told me that he proved this conjecture for all but finitely many graphs. 3. 96. In a more recent paper [YC1] Caro used a method from [YC] to get the best known bounds for Ramsey numbers. 9.96.*

[YC], Yair Caro, Colorability, Frequency and Graffiti-119, accepted for publication in Journal of Combinatorial Mathematics and Combinatorial Computation.

[YC1] Yair Caro, Books, Cycles and Ramsey, preprint 96.

120. *For every graph the independence number $\leq$ n-radius. This follows from Theorem 2 of [FA1]. It is interesting that radius provides also a lower bound for independence (comp. conj. 0).*

121. *If G is triangle-free then 2(number of edges) $\leq$ sum e(v), where e(v) is the number of vertices at even distance from v.*

Let D(k) be the number of pairs of vertices at distance k. If G is triangle-free then D(2) $\geq$ D(1)-n/2, s.f.

10. 96. See comment in conj -1 about relation to Ramseyan properties described in conjectures following 822. 10. 96.

Comp. 111, 240 and 245.

Paul Erdos *has conjectured and* **Zoltan Tuza, Computer and Automation Dept. of Hungarian Academy of Sciences** *proved that if G contains no* K_p *then* $D(2) \geq \frac{D(1)}{p-2}$ *- n/2. June 88.*

Let $e_t(G)$ *denote the number of nonedges x,y such that G+x,y contains* K_t *and let* $t(n,p)$ *be the number of edges in the p-partite critical Turan graph. Tuza proved that if G is* K_s *-free then*

e_s*(G $\geq$ D(1) + n(n-1)/2 - 2t(n,s-1). He suggests the problem of finding a function f(n,s,p) such that if G is Ks-free then*

$$e_s(G) \geq D(1)/(s-p+1) - f(n,s,p).$$

July 88.

If G is a graph then G' denotes the graph obtained from G by deleting a vertex of maximum degree. Repeating this operation we end-up with an independent set which will be called **Maxine.** *I was very surprised to find out that for all but one of about fifty graphs for which Graffiti knew the independence number, Maxine was a maximum independent set. I do not know what is the significance of that, but the only difference between the two invariants occured in a random graph with 40 vertices, $p = 0.5$. Accordingly I use now this algorithm as a try-out for the independence and the clique number. It is worth however to keep in mind that conjectures are about Maxine and sometimes it may result in new stronger theorems, namely that the algorithm produces an independent set satisfying the conjecture, compare for example 2 and 147, 0 and 247, or 250, (see also comments to 69 and 147.)*

8.96. I am still surprised about the performance of Maxine, but MIN - the algorithm which selects for a maximal independent sets a vertex of min degree, deletes its neighbors, etc, seems to perform even better. 8.96.

Let T be the set of vertices of G which are contained in a cycle. The vector indexed by elements of T and whose v-th component is the size of the smallest cycle containing v is called the **twister** *of G.*

D2 = D2(G) *is the graph with the same vertices as G, two being joined by an edge if their distance in G is 2. For a vertex v of G, let* **df(v)** *be the number of nonedges in the graph induced by the neighbors of v. The resulting vector is called the* **deficiency of G.**

Even Parity *is the vector indexed by vertices of even degree whose components are the corresponding degrees.* **Odd Parity** *is the vector defined similarly with* *respect to vertices of odd degree.*

Laplacian *is the matrix indexed by vertices of G, having the degree of the vertex v on the corresponding entry of the diagonal, -1, if the corresponding vertices are adjacent and 0 otherwise. The matrix is also sometimes called the admittance matrix.*

The **gravity** *of G is the matrix indexed by vertices of G whose (u,v)-th entry is 0 if u = v or there is no path joining u to v, and otherwise it is (1/(n-1))(deg(u)*deg(v)/d(u,v)), where deg(v) denotes the degree of v and d(u,v) is the distance from u to v.*

The **derivative** *V' of a vector V is obtained from V by first sorting it from the smallest component to the largest and then putting V'(i) = V(i+1) - V(i), i = 1..n. The separator of a matrix is is the difference between the largest and the second largest eigenvalue. The separator of a graph or simply the separator is the separator of its adjacency matrix.*

July 88.

122. *The average distance* $\leq$ *n / mean of coordinates of Maxine.*

123. *size/2* $\leq$ *the rank of the gravity matrix.*

124. *size/2* $\leq$ *the rank of Laplacian. Disproved by s.f.*

125. *the matching number* $\leq$ *the rank of the gravity matrix.*

126. *radius* $\leq$ *the number of negative eigenvalues of Gravity.*

Written on the Wall

127. *minimum degree* $\leq$ *n / average distance. (comp. 62)*

Mekkia Kouider, L. R. I. University de Paris-Sud and Peter Winkler, Bellcore *proved that the average distance* $\leq 3 + n/(1 + mindeg)$*. 6. 89.*

Let d denote the expected distance of two not necessarily distinct vertices of G. **Shi Ronghua, Department of Applied Mathematics, East China Institute of Technology, Nanjing,** *conjectures that d $\leq$ ((n+1)/ (1+ minimum deg)) - 1 and equality holds true iff G is complete or (n-2)-regular. Ronghua proved this conjecture under some additional conditions. 1. 90.*

He Wenji and Li Shuanghu, The Hebei Academy of Sciences, Shijiazuhng, China, *constructed a family of graphs and formidable formulas for the average distance in these graphs in terms of the minimum degree. Their graphs are counterexamples to the conjecture of Ronghua and they propose in its place a weaker conjecture, very similar to 127. [WS], 10. 90.*

4. 98. It seems that a related or perhaps exactly this paper was published in Chinese, see [BSR1]. 4. 98.

Mekkia Kouider *and* **Peter Winkler, Bell Labs** *improved their above result by lowering* 3 *to* 2*, [KW]. Some statements made in this paper about Graffiti are incorrect, but Peter told me that this will be taken care of. 4. 98.*

Peter Puget *finally settled this conjecture, i.e., they got rid of the summand 2. At first this may seem like not much of an improvement, but it is. Note that our proof with Bill Waller that the average distance is* $\leq 1+$ *the independence number is short and simple, but Fan Chung's proof of conjecture 2 is highly nontrivial. 4. 98.*

"from **Peter Winkler:**

In my paper with Mekkia Kouider, JGT 25 #1 (1997), we mistakenly credit GRAFFITI to Fajtlowicz and Waller, instead of just Fajtlowicz. ... Note also that one of the "flaws" we claim for Conjecture 62 (that it was made for graphs regular *of degree d, vice graphs of minimum degree d) was corrected in Conjecture 127, offered after we wrote the paper but before Siemion learned of our result."*

—Pete

7. 98. Conjecture 127 was placed on this list in July of 88. 7. 98.

Let me just add that Bill Waller did plenty for Graffiti by becoming the first human being to get PhD for working on a conjecture of a computer program. Concerning the "flaw", it is quite possible that 62 was actually printed by Graffiti in the form that av. dist $\leq$ n/min deg.

Here is the description of the second alleged "flaw" following an example described in Kouider and Winkler's paper: " Since this and similar constructions (some regular) yield $\nu > \frac{n}{d+1}$, Graffiti can hardly be expected to have conjectured to contrary. But a human would probably have conjectured that an n-vertex graph with minimum degree d has $\nu \leq \frac{n}{d+1} + c$ for some constant c, which we prove for $c = 2$."

Nevertheless conjecture 834 shows that this argument is also very questionable. As stated 834 says that

$\nu \leq 1 +$ the maximum temperature of the complement of G, i.e., that it is equivalent to $\nu \leq 1+ \frac{n-d-1}{d+1}$. Since the right side of the last inequality is equal to $\frac{n}{d+1}$, then in fact the program made exactly the conjecture which supposedly it was hardly expected to make. Conjecture 834 is false, but that of course must be the case in view of the above concerns.

7. 98. And of course it must be false to contradict Peter's reasoning. 7. 98.

This may be quite a good point against the fear of false conjectures. The latter issue came up after my talk at a recent DIMACS chemistry meeting. My position *was that it is fairly irrelevant whether a conjecture is true or false as long as it is interesting.*

5. 98. The problem of getting computer to make true statements is of course of great interest, but it is called automated theorem proving. On the second thought perhaps there are other possibilities or at least possibilities with a different flavor - this is a reminder to myself to write about it here (Marcus-Lopes).

These problems - conjecture making and validation of statements - are certainly not independent, and the former one, in my opinion, is more fundamental. Suppose that had computers which can prove theorems better than humans - how would they know what should they prove. I also believe that human theorem-proving is to a great degree based on invention of right conjectures.

I should also mention that sometimes - for example in elementary geometry - almost all conjectures of Graffiti seemed to be correct. 5. 98.

It would be of course of considerable interest to find an example and a reasonable argument of what computers can not do, particularly as far as mathematics is concerned. The day this will happen may be the best day for computer science since Turing solved Hilbert's Decision Problem. I might say "refuted" since the problem was stated as a conjecture similarly as was Hilbert's 10th problem.

As far as I am concerned, Penrose's arguments from "Shadows of the Mind" are questionable, but not because the formal proof contains an error. If that was all, the thesis itself might be still valid, and the idea might be still basically correct, similarly as it was the case with, say, Kempe's attempt of proving the 4-color conjecture.

Apart from pointing to Penrose's error, Putnam in his review of the book suggests that the issue is an empirical question. The main thesis of Penrose is that digital computers can not have human-like mathematical intuitions. I can't think of *a better scientific test of this issue than probing computer's abilities to make conjectures. If anyone can think of any other sound method of finding the answer, I would be very grateful for suggestions. As far as mathematics is concerned, this test is Turing-like enough for me, because I do not particularly care for computers (nor humans) which lie. I have plenty of understanding for humans (and computers) who make mistakes, because to start with, not much can done about it (in either case.)*

I have a candidate for a problem, which may prove to be undoable for computers, in which case this may seriously limit their abilities to do mathematics. It is the problem of inventing genuinely new concepts, by which I mean concepts which are not even implicitly present in the statement of the problem (or the representation of data.) There are claims about programs which can invent interesting mathematical concepts, but I am sceptical even about the meaning of such statements, because I wonder what are those mathematical concepts that are not interesting. I consider conjecture interesting, if it can inspire mathematicians to work on it, and I would agree that a concept is non-interesting if one can not make an interesting conjecture involving this concept. I think that there is good chance that Graffiti is capable of inventing on a short notice an interesting conjecture about any graph-theoretical concept, as long as it is easily computable - if it is not, then it make take longer. Of course, I may be mistaken, but I'll gladly accept any challenge to the effect.

Invention of genuinely new concepts, or concepts needed to solve problems, is a different story. I believe that this part of human mathematical skills is innately related to our linguistic abilities, so there is a possibility that the problem will prove to be as difficult as the natural language processing. That was my motivation for Invariant Interpolation Problems defined in 814. Perhaps working on these problems we may get ideas of how the process can be automated (or understood.)

Once on the subject of what computers can or can not do, I should mention here a conversation with Professor Simon a few years ago. He complimented me on the program after I reassured him that Graffiti indeed invented original conjectures of interest to mathematicians, but he was highly sceptical about my claims of the domain-independence of the program and in particular about its potential for making conjectures in chemistry and physics. Coincidentally, during my visit in Pittsburgh, he delivered a general lecture in which among others he argued that successful discovery programs must be domain-specific. I think the developments resulting from 840 and 841, particularly the paper of Patrick Fowler which explains the physical meaning of conjecture 841 favor my position. 4. 98.

Craig Larsen had within last few weeks several interesting thoughts concerning subjects discussed here. To explain the first one I have to mention an article, I once wrote on invitation of an editor of "Computers and Philosophy." The article was never published, because the referee had such absurd suggestions for changes, that I did not even write back. To illustrate the idea of Graffiti I invented there a planet where mathematicians do not prove theorems, but accept in their knowledge system all statements to which they do not have yet counter-examples.

I argued that on this planet bridges might hold as well as on ours. When I told this to Erdos, I've learned that he also invented a similar planet. On his planet there is a monument built for the first mathematician who suggested that one should prove the Riemann Hypothesis. When Craig learned about it, he told me that Putnam also conjured-up a planet of a kind in his article "What is Mathematical Truth?" [P1] (equally relevant seem the preceding article in the same volume: "Mathematics without Foundations". sf)

Putnam's planet is more like Erdos's than mine, because Putnam's mathematicians do quasi-empirical mathematics, i.e., they have no objections against formal methods; theorem proving is just a part of their repertoire.) On my planet, those who bring up the idea of the proof are routinely expelled from the Planet's Mathematical Society (and deported to Earth.)

Craig's second observation was that counter-examples in Graffiti could be viewed as heuristics to select better conjectures. This a nice way to express the action of the Dalamatian version which (sort of) wraps around new examples as if they were challenging the program to invent conjectures fitting these examples. I write about it in "Program Accelerators." The point is that unlike it is the case with the formal mathematics where counter-examples just prevent accepting some statements as true, on my planet, counter-examples help mathematicians to make new conjectures. Of course, I do not advocate doing away with theorem-proving, but primarily because I believe that the task of proving forces us to invent new concepts, and conjectures in terms of these new concepts reveal new properties of objects in which we are interested. I believe that this is the main epistemological function of theorem proving in mathematics. Working mathematicians do not have to be , and perhaps it is better if they are not, aware of this function, similarly as the basilisk dashing through conjecture 15, may be better of, without thinking about mechanics. 5. 98.

A month ago Raul Valdes-Peres quoted Noble-prize winner chemist, Roald Hoffmann on the subject of what computers can or can't do, in his letter to the "New Scientist." I found the book fascinating, particularly, the computational aspects of synthesis, and reading it, it occured to me that the problem of constructing of a molecule with desired properties is in sense quite similar to Invariant Interpolations

Problems. This may be somewhat cryptic - it is again a note to myself - but "dualizing" Dalmatians by taking the transpose of its database, one can get a program conjecturing objects, and in particular molecules.

I realized this duality from the very beginning, and I sometimes I think of formal structures represnting the program to express this duality, but I never could think of a second, interesting, down-to-Earth example. An example of self-dual situation are jigsaw puzzles. This could be stretched to proteins, but this is rather very far-fetched example as opposed to molecules which at certain level are simply representated by graphs.

The statement that Raul quotes is "the psychology of human beings is not well suited to admitting that we can be replaced by a computer program, only that others can be." Perhaps, it is the psychology, but I still think that the main problem is the language. It does not bother me the least, that Graffiti can make better conjectures than I do; what does bothers me, is that the program can't invent concepts. 7. 98.

Below is a letter from **Professor Simon.**

*"I don't recall the exact context of the issue of domain specificity, but let me try an answer to what I think is the question. Some of the discovery programs we have built, notably BACON, are wholly task independent – that is to say, when given a new task, they do not need to be given information specific to the task domain, or even a semantic interpretation of the problem. So, BACON given raw data shown pairs of values of x and y that "happen" to be exactly those that Kepler used in finding his Third Law, quickly (on the third try) finds that the y = ax**3/2 fits the data. The same program, without modification, finds Ohm's Law, Black's Law of temperature equilibrium of liquids, etc. etc. Its heuristics are completely general,*

not in the sense that they would solve all such problems (obviously!), but in the sense that they make no reference to the semantics of the domain of the data.

On the other hand, the KEKADA program, which designs sequences of experiments to find the solution to a scientific problem (e.g., the chemical path of urea synthesis in vivo, or the induction of electricity from magnetism), requires both general and specific knowledge. For example, it has a general procedure for exploiting surprises (but even there, recognition of a "surprise" is domain-specific). On the other hand, for the urea synthesis problem, it must know that if the final product contains nitrogen, then there must be nitrogen sources in the inputs – a fact of chemistry and physics.

This combination of the general and the task-specific should not surprise a mathematician. Mathematical induction is a rather general heuristic, used over a wide set of domains, and proof by contradiction perhaps even more general. On the other hand, Wiles spent seven years of his life developing specific methods and key lemmas in number theory to help him find his way to the proof of Fermat's Last Theorem. His success was absolutely dependent on domain-specific knowledge. Chess provides another example. The heuristic (not a very good one) "Always check, it may be mate!" is absolutely specific to that game, and makes no sense, even as a joke, in any other context. So are much more credible heuristics: "If you have an advantage in an open game, explore forcing moves first, including sacrifices." – also wholly domain-specific.

As both the Wiles and chess examples could show, it may be possible to conjure up "reasons" why the domain-specific heuristics work, but the reasons will themselves usually have a domain-specific as well as a general component, and will usually be as much empirical (i.e. dependent on experience) as logical. That is one *reason why no one becomes a world-class expert in any domain without ten years of intense study and practice in that domain – no universal experts!*

I am not sure that these remarks are relevant to our earlier exchange, but in any event, I hope they throw some light on my views about the respective roles of the general and the domain-specific in discovery."

Cordially

Herb Simon

Concerning the concept-formation problem Professor Simon wrote later in one of his letters:

"Before I could say much more about concept invention, I would have to know what you mean by "genuinely new concepts." I believe in the rule "garbage in, garbage out," but also in the rule "nothing in, nothing out." If someone knew about addition of integers, and already had the idea of the inverse of an operation and the idea of closure of an operation, and if s/he then arrived at the idea of negative numbers, would this be a "genuinely new concept"? Was "real number," defined by Dedekind's cut, a GNC? Do you have an example of a GNC that arose out of whole cloth, or do I misunderstand you?

I would take Darwinian evolution (or BACON, for that matter) as a model of the genuinely new arising from the old by combinatorics. Is there another way, even in mathematics? Didn't even God need some mud to make Adam, and a rib to make Eve?

But from your comment on natural language, I surmise that the line you are drawing may be less severe. I can point to work by Laurent Siklossy that I regard as a "proof of concept" of the process for learning natural language. But it requires an outside sensible world to provide the semantics – language, in this conceptualization, *is part of the physical world, not just the mathematical world. You will find a chapter by Siklossy in a book he and I edited (Representation and Meaning), published by Prentice-Hall in 1972.*

So please explain, so that I will not be tilting at windmills."

Cordially,

Herb

Defending my position, I quoted Conway's idea of numbers and games as an example of mathematical creation out of nothing (though really, out of the empty set.) The idea, by the way, was described and circulated by Cambridge students in a form of a biblical-style pamphlet. As far as computers are concerned they, indeed can recreate every mathematical idea, since they can simply print all conceivable sequences of symbols. Let's say that they can make light. My question is, how would they know that the "light was good."

The most novel aspect of Dedekind cuts and Conway numbers is not just the simplicity nor parsimony of their ideas. These ideas are novel, because they throw a new light on the on what numbers are, or to be more precise, on how one can think of them. I indeed can not say clearly what would make a computer-invented concept genuinely new, but I have the same problem with human-invented concepts. On the other hand, if computer will invent a GNC, then its novelty and significance should be (or at least should eventually become) obvious. Perhaps instead of calling them GNC I should have use the term: "the difference making concepts." 8.98.

Concerning my statement that "it is fairly irrelevant whether a conjecture is true or false as long as it is interesting." **Edward Kirby** *wrote: " I agree with that, but is it not in part dependent on achieving some (real but unquantified) statistical level of success that is acceptable? If you found that almost ALL the conjectures* *generated turned out to be false, would they still be as interesting? On second thoughts, you do say "fairly" irrelevant!"*

I guess that this problem may look completely different from the point of view of mathematicians whose principal goal of is to prove statements as opposed to other scientists who are primarily interested in just knowing the truth. If a false conjecture leads someone to a proof of an interesting statement T then in effect the conjecture is as good as conjecturing of T. Quite a few of conjectures of Graffiti resulted in proofs of the same claims under somewhat stronger assumptions and first of all, from the point of the program, counterexamples still may be more essential (see above) than proofs of conjectures. Until I began to use the program as accelerator, the proofs did not matter at all (for the program). The program aside, I think that counter-examples (or simply examples) play much more significant role in mathematics (and other science too) than it is generally acknowledged. This aspect is difficult to appreciate, because examples are usually easier to find than proofs; easier does not necessarily means less significant.

In principle, it is possible to have systems which make interesting conjectures most of which are false, but this seems to me extremely unlikely. The problem with Graffiti, was exactly opposite. Most of its, particularly early, conjectures were true, actually trivially true, i.e. , completely non-interesting from mathematical point of view.

Concerning concepts which are not even implicitely present in statement of the problem, Kirby wrote:

"It seems to me that there are difficulties in the term 'implicit'. I suppose that in practice it comes to be accepted that Concept B is genuinely new if no one succeeds in demonstrating that it arises logically in some manner from the statement

of Problem A, but can one ever prove this (that it cannot be done) or be sure that a connection will not be brought out at a later date by some thinker looking at the matter afresh?

One can of course simply call the concept new if, in fact, it has occurred to no one else before.

Besides this element of subjectivity in the word, I feel there is somewhere a lurking philosophical difficulty, in that, since we presume we are living in one universe that is coherent and rationally constructed, can any concept NOT be implicit in an ultimate sense?"

My point of view is close to the next to the last statement. Concerning the last one, I just have this reflection that the appearance of coherence or rationality of the universe (or any other paradigm) depends very much on concepts we invent to understand it.

[BRS1] Robert A. Beezer, John Riegsecker and Bryan A. Smith, Using Minimum Degree to Bound Average Distance, preprint '98.

[C] John Horton Conway, On Numbers and Games, Academic Press, '76.

[H] Roald Hoffman, The same and not the same, Columbia University Press, '95.

[KW] Mekkia Kouider and Peter Winkler, Mean Distance and Minimum Degree, JGT, 1997, pp. 95 - 99.

[LS] Pat Langley, Herbert Simon, Gary L. Bradshaw, and Jan M. Zytkow, Scientific Discovery, The MIT Press, 1987.

[P] Hillary Putnam, The New York Times Book Review, Nov.20,1994, p.7.

[P1] Hillary Putnam, Mathematics, Matter and Method, Philosophical Papers vol 1.

[P2] K. Popper, "Conjectural Knowledge," in Objective Knowledge,

128. *The second smallest eigenvalues of Laplacian* $\leq n/averagedistance$.

Vance Faber, Los Alamos National Laboratory, *deduced from the results of [CF] that for every fixed d = maximum degree there are at most finitely many counterexamples to this conjecture and even to the statement that diameter* $\leq$ *n/ largest eigenvalue of Laplacian. He also proved that if 0.5 ln(n-1)*largest eigenvalue of Laplacian* $\leq$ *n then the conjecture is true. 7. 90.*

[CF] Fan Chung and Vance Faber

129. *deviation of eigenvalues of Laplacian* $\leq$ *the Randic index.*

130. *range of the deficiency* $\leq$ *n - the matching number.* [**FMS1**]. *December 88.*

131. *minimum deficiency* $\leq$ *size / average distance.*

132. *The mean deficiency* $\leq$ *size - the matching number. The equality holds true iff G is a matching. [FMS1]. December 88.*

133. *Sum of reciprocals of components of twister* $\leq$ *harmonic.*

134. *The Randic index* $\leq$ *the rank of gravity.*

135. *chromatic number / clique* $\leq$ *independence of D2. S. F. 7.89.*

136. *Deviation of Temperature* $\leq$ *Randic.*

137. *2nd largest eigenvalue* $\leq$ *harmonic.* **James B. Shearer** *October 88. [FMS2]. November 88.*

138. *2-nd largest eigenvalue* $\leq$ *size / clique.* [**FMS2**] *proved that this conjecture is true for all graphs but* K_2 *.December 89.*

139. *- (2-nd smallest eigenvalue)* $\leq$ *harmonic.* **James B. Shearer.** *October 88.*

140. *deviation of eigenvalues $\leq$ harmonic.* **FMS**. *10.89.*

141. *range of positive eigenvalues $\leq$ the matching number.* [**FMS2**]. *December 88.*

142. *minimum positive eigenvalue $\leq$ n / average distance.*

143. *variance of positive eigenvalues $\leq$ size / average distance.*

144. *variance of positive eigenvalues $\leq$ size - matching.*

145. *minimum of derivative of positive eigenvalues $\leq$ n/average distance. [FMS2], December 89.*

146. *The sum of positive eigenvalues $\leq$ size. The equality holds true iff the maximum degree is 1.* **James B. Shearer**, *July 88.*

147. *The average distance $\leq$ the number of vertices whose coordinates of Maxine is 0.*

With a right (or really wrong) ordering of vertices in barbell graphs, [FA] , Maxine may select every third vertex in the central path, so the strongest version of the observation is false. S. F. September 88. See also 212 and 246.

148. *The average distance $\leq$ harmonic.*

149. *mean of gravity $\leq$ sum of components of D, where D is the vector whose vth component is the number of vertices of odd distance from v. Siemion Fajtlowicz, August 88.*

150. *The minimum of derivative of eigenvalues of the gravity matrix is $\leq$ n/average distance.*

151. *The number of positive eigenvalues of gravity $\leq$ than the matching number.*

152. *max(range of Even Parity, range of Odd Parity) $\leq$ n / average distance.*

153. *The range of Odd Parity $\leq$ the matching number. S. F. August 88.*

154. *deviation of eigenvalues $\leq$ n / average distance.*

155. *mean of autocoordinates of Maxine of D2 $\leq$ the matching number.*

156. *mean of autocoordinates of Maxine of D2 $\leq$ the chromatic number.*

* * *

157. *radius $\leq$ min (min Odd, min Even).* **s.f.**, *July 88.*

* * *

158. *the independence number of a graph is not more than the number of vertices - minimum degree.*

Tony Brewster, Los Alamos National Laboratory, *proved that the equality holds true iff the complement of the graph has a component that is clique, and every other component has maximum degree less than the number of vertices of this component. 6. 91.* **Tony Brewster, Michael Dineen and Vance Faber, LANL** *tested in '90 -'91 about 200 conjectures of Graffiti against all 10-vertex graphs (about 12 000 000 graphs) and about 1/3 of the conjectures they tested proved to be false.*

9. 86. After they completed their tests (of about 200 conjectures) the number of false conjectures was closer to 1/5th. 9.86.

[BDF], Computational Attack on Conjectures of Graffiti, Discrete Mathematics, 147 (1994) 35 - 55.

Conjectures for triangle-free graphs.

159. *size / clique $\leq$ sum of the vector E from 96 . From the result stated in comments to conj. 121 it follows that sum of Even size / (clique-1).*

July 23. 88

160. *- smallest eigenvalue $\leq$ size/2.* **Odile Favaron, Maryvonne Maho and Jean-Francois Sacle** *notice that largest eigenvalue-smallest eigenvalue $\leq$ size/2 and the equality holds true iff G is a a union of a complete bipartite graph and isolated vertices. December 88.*

161. *The maximum of autocoordinates of Maxine of the complement of $G \leq$ the average transmission of the distance matrix.*

162. *chromatic number / clique $\leq$ the range of positive eigenvalues.*

163. *chromatic number / clique $\leq$ minimum of Even.* [**FMS1**]. *November 88.*

164. *mode of the degree sequence $\leq$ size / average distance.* **James B. Shearer**. *10. 89.*

165. *mode of eigenvalues of Laplacian $\leq$ size / average distance.* **Tony L. Brewster, Michael Dinneen and Vance Faber**, *see 107 and 158.10. 90.*

166. *size/2 $\leq$ the number of nonpositive eigenvalues of the distance matrix,* **Tony L. Brewster, Michael Dinneen and Vance Faber**, *see 107. 7.91.*

167. *The sum of reciprocals of eigenvalues of Laplacian $\leq$ the mean transmission of the distance matrix.*

proved in "Average distance in graphs and Eigenvalues" by **Sivaramakrishnan Sivasubramanian School of Technology and Computer Science Tata Institute of Fundamental Research.**

168. *The minimum of derivative of eigenvalues of Laplacian $\leq$ n/independence number.*

169. *The Randic index $\leq$ rank of Laplacian. S. F, 9. 88.*

170. *deviation of temperature $\leq$ size / clique.*

171. *deviation of temperature $\leq$ n / average distance.*

172. *minimum of derivative of eigenvalues $\leq$ n / independence.*

173. *n / average distance* $\leq$ *length of eigenvalues of Laplacian.*

174. *The sum of reciprocals of vector E from 96 is* $\leq$ *n/ average distance.*

175. *mean of autocoordinates of Maxine of D2* $\leq$ *n / average distance.*

Conjectures for connected graphs (176:180)

176. *n / independence* $\leq$ *sum of all temperatures.* **[FMS1].** *November 88.*

177. *range of Even Parity* $\leq$ *size/2. S. F. September 88.*

178. *- 2-nd smallest eigenvalue* $\leq$ *the matching number.* **James B. Shearer**, *October 88.*

179. Inverse transmission of Gravity average transmission of Distance. **Tony L. Brewster, Michael Dinneen and Vance Faber** *(comp. 107), 10. 90. Interestingly, the only counterexample found was K2. It is difficult to tell now whether this example was missing at the time, or whether the conjecture was result of a bug.*

180. *The sum of reciprocals of the the components of vector E from 96 is* $\leq$ *the chromatic number.* **[FMS1]**

Conjectures for connected graphs in which the sum

of components of D is $\leq$ *the sum of components of E (181:204)*

where E and D are vectors defined in 96.

July 26, 88.

181. *The range of resolution of Maxine* $\leq$ *maximum of E.*

182. *The mean of autocoordinates of Maxine of the complement of G* $\leq$ *size/average distance.*

183. *The length of autocoordinates of Maxine of the complement of G* $\leq$ *the mean transmission of the distance matrix.*

184. *The maximum of autocoordinates of Maxine of the complement of G* $\leq$ *n - the matching number.*

185. *size / independence* $\leq$ *length of the degree sequence.* **Odile Favaron, Maryvonne Maheo and Jean-Francois Sale**. *December 89.*

186. *size/independence* $\leq$ *the sum of absolute values of eigenvalues.* **Odile Favaron, Maryvonne Maheo and Jean-Francois Sale**. *December 89.*

187. *The mode of eigenvalues of Laplacian* $\leq$ *n - the independence number.***Tony L. Brewster, Michael Dinneen and Vance Faber** *(comp. 107), 10. 90.*

188. *The mode of eigenvalues of Laplacian* $\leq$ *n - the matching number.* **Michael J. Dinneen, Los Alamos National Laboratory and University of Victoria, Victoria, B.C** *(comp. 107.) August 91.*

189. *The mode of eigenvalues of Laplacian* $\leq$ *number of nonpositive eigenvalues.* **Tony L. Brewster, Michael Dinneen and Vance Faber** *(comp. 107), 10. 90.*

190. *deviation of eigenvalues of Laplacian* $\leq$ *mean of* E. **Odile Favaron, Maryvonne Maheo and Jean-Francois Sacle**. *December 89.*

190. *The deviation of eigenvalues of Laplacian* $\leq$ *mean of* E. **Odile Favaron, Maryvonne Maheo and Jean-Francois Sale**. *December 89.*

191. *The minimum deficiency* $\leq$ *size / clique.* **Odile Favaron, Maryvonne Maheo and Jean-Francois Sale**. *December 89.*

192. *The mean of Even Parity* $\leq$ *n - the matching number.* [**FMS**]. *October 88.*

193. *The mean of Odd Parity* $\leq$ *n - the matching number.*[**FMS**]. *October 88.*

194. *The maximum eigenvalue* $\leq$ *size / average distance.* **Odile Favaron, Maryvonne Maheo and Jean-Francois Sacle**. *December 89.*

195. *The maximum eigenvalue* $\leq$ *maximum of* E.

196. *2-nd largest eigenvalue* $\leq$ *mean of* E. **[FMS2]**. *November 88.*

197. *- 2-nd smallest eigenvalue* $\leq$ *range of eigenvalues of the gravity matrix.*

198. *minimum of derivative of eigenvalues* $\leq$ *n / mean gravity.*

199. *- smallest eigenvalue* $\leq$ *mean of* E. **[FMS2]**. *November 88.*

200. *The minimum positive eigenvalue* $\leq$ *the Randic index.* **Odile Favaron, Maryvonne Maheo and Jean-Francois Sacle** *, proved that the conjecture is correct for all but certain for complete p-partite graphs. January 89.*

201. *The minimum of derivative of positive eigenvalues* $\leq$ *size/clique.* **[FMS2]**, *November 88.*

202. *The average distance* $\leq$ *maximal frequency of the degree sequence.* **Peter Puget,** *11, 89.*

203. *The average distance* $\leq$ *the sum of reciprocals of degrees.* **Vance Faber, Los Alamos National Laborarory,** *8,90.*

Conjectures for connected graphs in which the sum of components of E is $\leq$ the sum of components of D (204:211)

where D and E are vectors defined in 96.

July 26, 88.

204. *The length of autocoordinates of Maxine of D2* $\leq$ *the mean transmission of the distance matrix.*

205. $n/2 \leq n$ *- the residue.* **[FMS]**. *October 88.*

206. *2-nd largest eigenvalue* $\leq$ *the matching number.* **James B. Shearer**, *October 88.*

207. *- smallest eigenvalue* $\leq$ *the matching number.* **[FMS2]**. *October 88.* **James B. Shearer**, *October 88.*

208. *- smallest eigenvalue* $\leq$ *harmonic.* **James B. Shearer**, *October 88.*

209. *The sum of positive eigenvalues* $\leq$ *the mean of the transmission of the distance matrix.* **James B. Shearer**, *88.*

210. *The average distance* $\leq$ *number of negative eigenvalues of the gravity matrix*

211. *n / average distance* $\leq$ *the sum of absolute values of S. F. 2, 90.*

Conjectures for triangle-free graphs (212:220)

212. *Inverse coordinates of Maxine* $\leq n/2$*.*

5-vertex path is a counterexample if the vertices are ordered in a right manner. James B. Shearer, August 88. Graffiti knows this graph and it would be nice to have an example defeating Maxine independently of the representation of the graph. See also comments about Metromaxine. I think that the conjecture should be true for maximum independent sets. Somewhat more generally one can ask: are there and and if yes , what are the smallest constants c_p *such that if G is* K_p*-free then the inverse coordinates of a maximum independent set* $\leq c_p \times n$ *Favaron, Maheo and Sacle proved that* c_p *=(p-2)/(p-1).[FMS1]. October 88.*

Def. a graph is **mean** *if every maximum independent set J contains all vertices of degree greater or equal to the average degree.*

If G is mean than Maxine can't produce a maximal independent set in G, actually it is enough if every J contains all vertices of maximum degree.

Bipartite graphs are not mean, because the smaller independent set emanates the same number of edges as the larger and thus contains a vertex of degree not less than the average.

Is there a triangle-free mean graph? **Paul Erdos and Ralph Faudree, Memphis State University**, *constructed the following examples of mean graphs: G is* *the join of D_k and M_l where D_k has k vertices and no edges,M_l is a matching of l disjoint edges, and 2l > k > l. November 88.*

213. *size/2 $\leq$ Randic.* [**FMS2**] *. November 88.*

214. *size / independence $\leq$ n - independence, August 88. This can be generalized as follows: let G^* denotes a maximum K_{p-1}-free induced subgraph of G and let b be its number of vertices. If G is K_p-free, p $\geq$ 3, then the number of edges of G is not more then $t_{p-1}b$ + b (n-b). The proof is very much like that of Erdos majorization Theorem and as matter of fact we have Theorem. If G is K_p-free then G is degree-majorized by the join of G^* and D where D has n-b vertices and no edges. What is the maximum number of edges of K_p-free graph G in terms of G^* if G^* denotes now a maximum K_q-free induced subgraph of G? What is the order of G^*? As q decreases from p to 2 these questions shift from Turan-type to Ramsey-type problems. August 88.*

215. *size / independence $\leq$ scope of the eigenvalues.*

Counterexamples are graphs obtained by removal of edges from triangles in random graphs in which n >> average degree >> 1.They are discussed in [SH1]. These graphs can be used to obtain counterexamples of arbitrarily large girth and one can similarly construct counterexamples to 241 and 243 . **James B. Shearer**, *October 88.*

216. *size / independence $\leq$ mean of Even.* **James B. Shearer**. *October 88.*

217. *deviation of Temperature $\leq$ n / mean Gravity.*

218. *maximum eigenvalue $\leq$ size1/2.* **FMS** *pointed to me that this is a result of Nosal, [CDS], p. 89. Proved independently by* **James B. Shearer**. *October 88.*

219. *2-nd largest eigenvalue of Gravity $\leq$ number of nonedges.*

220. *maximum of Odd $\leq$ matching. s.f. August 88.*

August 3, 88.

Conjectures for graphs of girth $\geq$ 5.

221. *radius $\leq$ sum of reciprocals of the degree sequence.*

222. *size / independence $\leq$ n/ average distance.*

223. *2-nd smallest eigenvalue of Laplacian $\leq$ n / independence.*

224. *harmonic $\leq$ rank.*

225. *average distance $\leq$ residue.*

226. *average distance $\leq$ n / mean Gravity.*

Conjectures for regular graphs (227:239)

The vectors D and E are defined in 96.

August 4, 88

227. *diameter $\leq$ matching.* [**FMS1**] *constructed cubic graphs in which diameter is approximately 3n/4 and matching is approximately n/2, this is probably the worst possible case. September, 88. Also disproved independently by* **James B. Shearer**, *October 88.*

228. *mean of coordinates of resolution of Maxine $\leq$ size/2.*

229. *maximum of coordinates of resolution of Maxine $\leq$ the matching number.*

230. *minimum of autocoordinates of resolution of Maxine of the complement of G $\leq$ n / independence.*

231. *If G is a regular connected graph then chromatic number of G is not more than n/average distance.*

This conjecture follows from (still unproved) 62, comp also 127. 9. 97.

232. *size/2 $\leq$ n- independence.*

and the equality holds true iff G is $K_{n,n}$. **James B. Shearer**, *August 88.*

233. *maximum of the derivative of eigenvalues of Laplacian* $\leq$ *n / average distance.*

234. *n - rank* $\leq$ *size / average distance.* [**FMS1**]. *November 88.*

235. *average distance* $\leq$ *minimum of* D.

236. *average distance* $\leq$ *minimum of* E.

237. *The sum of reciprocals of* $D \leq n$ */ average distance.*

238. *The mean of autocoordinates of resolution Maxine of the complement of* $G \leq n/2$.

239. $n/2 \leq$ *the maximal frequency of* E.

Conjectures for K_4-free graphs (240 : 245).

240. *size-n* $\leq$ *sum of Deficiency.* **Odile Favaron, Maryvonne Maheo and Jean-Francois Sacle**. *The equality holds true iff G is a triangle. December 88. Comp 245. [FMS1].*

241. *size / independence* $\leq$ *scope of eigenvalues. Disproved by* **James B. Shearer,** *see his solution of 215. October 88.*

243. *size / independence* $\leq$ *maximum eigenvalue of Laplacian. Disproved by* **James B. Shearer**, *see his solution of 215. October 88.*

244. *deviation of eigenvalues of Laplacian* $\leq n/2$. **FMS** *12.88.*

245. *size* $\leq$ *sum of Even.* **Odile Favaron, Maryvonne Maheo and Jean-Francois Sacle**. *December 88. Comp 240, 111 and 121. [FMS1].*

In conjectures below matchings and the chromatic number are computed by greedy algoritms. The partition produced by the algorithm for the chromatic number is called the **coloration**. *The* **rainbow** *of a partition is the vector indexed by vertices of G whose component corresponding to the vertex v is the number of equivalance classes containing a vertex adjacent to v. Rainbow of the coloration will* *be simply refered to as the* **rainbow**. *The* **dual degree** *is the vector indexed by vertices of G whose component corresponding to the vertex v is the mean of degrees of the neighbors of v. This concept was studied in [Sh1].*

Conjectures for arbitrary graphs (246 : 274).

246. *The radius $\leq$ number of zero coordinates of Maxine.*

If G is cycle of order $6k$ then Maxine may find an independent set of order $2k$, which is also a counterexample to 147. **FMS**. *September, 88.*

247. *Radius is not more than sum of reciprocals of rainbow. The strongest interpretation of this conjecture was disproved by* **Ermelinda DeLaVina**. *1. 91.*

248. *The number of zero coordinates of Maxine $\leq$ chromatic number of complement of G.*

This conjecture is obvious if the invariant on the left is interpreted as a maximum independent set.

249. *Range of rainbow is not more than the chromatic number of the complement of G.*

*The strongest interpretation of this conjecture is false but we do not know an example of a graph in which every coloration would be a counter-example. (***Ermelinda DeLaVina and S.F**. *1.91.*

250. *The minimum of rainbow of the complement of $G \leq$ the matching number. Disproved by* **Ermelinda Delavina, University of Houston**. *1. 91.*

251. *The minimum of the dual degree $\leq n$ / average distance.* **[FMS1].** *November 88.*

252. *The minimum of derivative of eigenvalues of Laplacian $\leq$ the sum of reciprocals of the dual degree.*

253. *The deviation of eigenvalues of Laplacian* $\leq$ *the maximum of the dual degree.* **Odile Favaron, Maryvonne Maheo and Jean-Francois Sale**. *December 89.*

254. *The minimum of derivative of eigenvalues of Laplacian* $\leq$ *the sum of reciprocals of Odd.*

255. *The Randic index* $\leq$ *sum of coordinates of Maxine.*

256. *Maximum eigenvalue of a graph is not more than its maximum dual degree. The* ***dual degree*** *of a vertex is the mean of the degrees of its neighbors. This conjecture was proved by* **Shearer**. *Here is his elegant argument. Let D be the diagonal matrix with degree of the vertex as d[i,i] entry, and let A be the adjacency matrix. Then* $D^{-1}AD$ *has the same eigenvalues as A, but its row-sums are dual degrees. This generalizes the well known result that the largest eigenvalue is not more than the maximum degree. 10.88.*

Favaron, Maheo and Sacle *found independently the same proof and they noticed the proof implies that the equality holds true iff every vertex has the same dual degree. They found many examples of nonregular graphs, for which the equality holds true. 11. 88.*

258. *The number of positive eigenvalues* $\leq$ *the matching number + the matching number of the complement.* [**FMS2**] . *November 88.*

259. *The mean of autocoordinates of Maxine of* $D_2 \leq$ *the matching number of the complement.*

260. *The matching* $\leq$ *the size - the scope of Even Parity .* **Odile Favaron, Maryvonne Maheo and Jean-FrancoisSacle**. *November 88. They proved a somewhat stronger results and in both cases they characterized cases of equality. [FMS1].*

261. *The matching* $\leq$ *size - the sum of reciprocals of the even parity,* **[FMS1]** *August 21, 88*

262. *-smallest eigenvalue* $\leq$ *maximum of Even.*

263. *range of coordinates of Maxine* $\leq$ *1 + range of positive eigenvalues.*

264. *2 + range of positive eigenvalues* $\leq$ *bichromatic number. Michael J. Dinneen, Los Alamos National Laboratory and University of Victoria, Victoria, B.C (comp. 107.) August 91.*

265. *2 - smallest eigenvalue* $\leq$ *bichromatic number. Odile Favaron, Maryvonne Maho and Jean-Francois Scle. December, 89.*

266. *mean of coordinates of Maxine* $\leq \frac{size}{2(av.dist)}$.

267. $\frac{lenght of Dual Degree}{2} \leq$ *bichromatic number. Odile Favaron, Maryvonne Maheo and Jean-Francois Sacle. December 89.*

268. $\frac{(averagedistance)(maximumdegree)}{2} \leq n$. *[FMS1]. November 88.*

269. $\frac{(averagedistance)(sumoftemperatures)}{2} \leq n$.

270. $2n^{1/2} \leq$ *bichromatic number. Odile Favaron, Maryvonne Mahe oand Jean-Francois Sacle. The equality holds true iff* $n = s^2$ *and either G or its complement is a union of s disjoint copies of* K_s *[FMS1]. December 88. I found however that this is a known result due to Fink, see [CM] p. 5.*

271. *(mean Gravity) (mean distance)* $\leq$ *sum of degrees.*

275. *If girth is* ≥ 5 *then radius* $\geq$ *sum of inverses of dual degrees.*

276. *If girth is* ≥ 5 *then the mean of coordinates of Maxine* $\leq$ *radius.*

277. *If girth is* ≥ 5 *then the mean of coordinates of Maxine* $\leq \frac{n}{independence}$.

278. *If girth is* ≥ 5 *then the mean of coordinates of Maxine* $\leq$ *the average distance.*

279. *If girth is* ≥ 5 *then the matching number* $\leq$ *the sum of inverses of the rainbow.*

280. *If girth is* ≥ 5 *then the mean of coordinates of mismatching* $\leq$ *matching.*

281. *max (matching, mismatching) number of negative eigenvalues of Distance. Michael J. Dinneen, Los Alamos National Laboratory and University of Victoria, Victoria, B.C (comp. 107.) August 91.*

282. *If girth is* ≥ 5 *then the n - the independence number* $\leq$ *rank of the distance matrix.*

283. *If girth is* ≥ 5 *then the independence* $\leq$ *number of nonpositive eigenvalues of the distance matrix.*

284. *If girth is* ≥ 5 *then the minimum dual degree* $\leq$ *- the smallest eigenvalue of distance matrix.*

285. *If girth is* ≥ 5 *then the sum of inverse of dual degrees* $\leq$ *the maximal frequency of Even. FMS 10. 89.*

286. *If girth is* ≥ 5 *then the second largest eigenvalue of Laplacian* $\leq$ *the matching + the matching number of the complement.*

287. *If girth is* ≥ 5 *then the second smallest eigenvalue of Laplacian* $\leq$ *the sum of inverses of dual degrees.*

288. *If girth is* ≥ 5 *then the mean deficiency* $\leq$ *the matching of the complement. FMS 10. 89.*

289. *If girth is* ≥ 5 *then the second largest eigenvalue* $\leq$ *the mean dual degree. James B. Shearer, October 88.*

290. *If girth is* ≥ 5 *then the - 2nd smallest eigenvalue* $\leq \frac{size}{meangravity.}$

291. *If girth is* ≥ 5 *then the scope of positive eigenvalues* $\leq$ *the* $\frac{size}{meangravity.}$

292. *If girth is* ≥ 5 *then the minimum positive eigenvalue* $\leq \frac{n}{mean gravity.}$

293. *If girth is* ≥ 5 *then the minimum of derivative of positive eigenvalues* $\leq \frac{size}{independence.}$ *Tony L. Brewster, Michael J. Dinneen and Vance Faber, (comp 107) They found eight counterexamples all of which were trees. February 91.*

294. *If girth is* ≥ 5 *then* $\frac{n}{average distance} \leq$ *chromatic number + chromatic number of the complement.*

295. *If girth is* ≥ 5 *then the number of positive eigenvalues of the distance matrix* $\leq \frac{n}{mean gravity.}$

August 25, 88.

296. *If G is a tree then the average distance* $\leq \frac{n}{range of coordinates of matching.}$ *Michael J. Dinneen, Los Alamos National Laboratory and University of Victoria, Victoria, B.C (comp. 107.) August 91.*

297. *If G is a tree then the second smallest eigenvalue of Laplacian* $\leq \frac{n}{independence.}$ *Odile Favaron, Maryvonne Maheo and Jean-Francois Sale. 2. 90.*

298. *If G is a tree then the n - number of endpoints* $\leq$ *matching + the matching number of the complement. Odile Favaron, Maryvonne Maheo and Jean-Francois Sacle proved that bimatching -1 is enough, and that equality holds true iff G is K2 or a path of an odd order. October 88. [FMS1].*

300. *If G is a tree then the (mean distance) (mean gravity)* $\leq$ *n-1.*

301. *If G is a tree then the scope of positive eigenvalues* $\leq$ *harmonic. James B. Shearer, October 88.*

302. *If G is a tree then the scope of positive eigenvalues* $\leq$ *the mean dual degree. James B. Shearer, October 88.*

August 26, 88.

303. *If the distance rank is strictly less than the rank then the radius* $\leq$ *harmonic.*

304. *If the distance rank is strictly less than the rank then the mean of coordinates of Maxine* $\leq$ *radius.*

305. *If the distance rank is strictly less than the rank then the sum of inverses of dual degrees* $\leq$ *the number of nonnegative eigenvalues. Tony L. Brewster, Michael J. Dinneen and Vance Faber, (see107) 12. 90.*

306. *If the distance rank is strictly less than the rank then the sum of inverses of dual degrees* $\leq$ *number of nonpositive eigenvalues.Tony L. Brewster, Michael J. Dinneen and Vance Faber, (see107) 12. 90.*

307. *If the distance rank is strictly less than the rank then the average distance* $\leq \frac{n}{largesteigenvalue}$*.Tony L. Brewster, Michael J. Dinneen and Vance Faber, (see107) 12. 90.*

308. *If G is a connected graph in which the rank of the distance matrix is strictly less than the rank then the average distance of G is not more than the residue.*

309. *If G is a connected graph in which the rank of the distance matrix is strictly less than the rank then the sum of inverses of Odd* $\leq$ *the matching number.*

August 27, 88.

Conjectures for triangle-free graphs.

* * *

310. *If G is a triangle-free graph then the* $\frac{size}{independence \leq}$ *harmonic. Siemion Fajtlowicz, August 88.*

* * *

311. *If G is a triangle-free graph then n independence* $\leq$ *range of coordinates of Maxine.*

312. *If G is a triangle-free graph then* $\frac{size}{independence} \leq$ *number of nonnegative eigenvalues.*

313. *If G is a triangle-free graph then* $\frac{size}{independence} \leq$ *number of negative eigenvalues of the distance matrix.*

314. *If G is a triangle-free graph then* $\frac{size}{independence} \leq$ *mode of Even. James B. Shearer, October 88.*

315. *If G is a triangle-free graph then minimum of Rainbow* $\leq$ *radius.*

316. *If G is a triangle-free graph then the chromatic number* $\leq$ *range of eigenvalues of Laplacian.*

317. *If G is a triangle-free graph then mode of the degree sequence* $\leq$ *minimum of Even. [FMS!]. November 88.*

318. *If G is a triangle-free graph then the maximum degree* $\leq$ *mode of Even. James B. Shearer, October 88.*

319. *If G is a triangle-free graph then the maximum of derivative of eigenvalues of Laplacian* $\leq$ *Inverse Rainbow.*

320. *If G is a triangle-free graph then* $n - residue \leq$ *the matching of the complement of G + the matching number of G.*

321. *If G is a triangle-free graph then the maximum eigenvalue* $\leq$ *mean of Even. James B. Shearer, October 88.*

322. *If G is a triangle-free graph then the Inverse Even* $\leq$ *range of eigenvalues of Distance.*

323. *If G is a triangle-free graph then the scope of positive eigenvalues* $\leq$ *mismatching.*

324. *If G is a triangle-free graph then the mean of Odd $\leq$ Inverse Rainbow.*

325. *If G is a triangle-free graph then the mean of Odd $\leq$ the matching of the complement of G. [FMS!]. December 88.*

326. *If G is a triangle-free graph then the mean gravity $\leq$ sum of coordinates of Maxine.*

345. *Cvetkovic deduced from Cauchy's Interlacing Theorem that for every graph the independence number is smaller or equal to the number of nonnegative eigenvalues as well as the number of nonpositive eigenvalues. He also noticed that there are graphs for which we have equality, but Graffiti database shows that there are quite a few such graphs. They are called here* ***plants****. The simplest examples are trees, because for these graphs the number of positive eigenvalues is the matching number, and for bipartite graphs the matching number + the independence number is the number of vertices. Plants in which the independence number is equal to to the number of non-negative eigenvalues are called* ***heliotropic*** *plants, and the other are called* ***geotropic****. Perhaps the most interesting conjecture on the subject was that PR[2..n] graphs (conj. 446) are heliotropic plants. The conjecture proved to be false, but it seems that in these graphs the independence number, which is equal to the number of primes less or equal to n is close to the number of non-negative eigenvalues.*

Conjecture. *If G is a plant then the average distance of G is not more than the sum of reciprocals of degrees of G.*

346. *If G is a plant then the average distance of G is not more than the number of distinct eigenvalues of the distance matrix of G. (345.)*

347. *let e be the number of edges of G, and c its clique number. If G is a plant then e/c is not more than the mean of row sums of the distance matrix of G.*

348. *The gravity matrix is indexed by vertices of G. The entry corresponding to the pair (u,v) is 0 if u=v or if they are in different components of G, and otherwise it is* $1/(n-1)deg(u)deg(v)d^{-1}(u,v)$, *where d is the distance between u and v.*

Conjecture: Let G be a plant and let E be the sorted vector of eigenvalues of G. Then min e(k+1) - e(k) is not more than the mean entry of the gravity matrix.

351. *If G is a heliotropic plant then the radius is not more than the number of positive eigenvalues.*

1 + the number of positive eigenvalues follows from the Interlacing Theorem, because graph of radius r contains an induced path of length 2r-1, [ESS].

352. *If G is a heliotropic plant then the matching number of G is not more than its chromatic number + the chromatic number of the complement of G.*

356. *If G is a geotropic plant then the radius is not more than the number of the negative eigenvalues.*

360.*If G is a geotropic plant then* $\frac{n}{independence} \leq$ *range of coordinates of Maxine.*

362. *If G is a geotropic plant then 1/2 (average distance) (the average degree)* $\leq$ *independence number.*

This conjecture was refuted by AutoGraphiX with a counter-example with 17 vertices. First AGX explored systematically graphs with up to 14 vertices then extrapolated the low regions of the surface so obtained, exploring them and finally using the interactive routine.

Pierre Hansen, 6. 98.

September 6, 88.

Conjectures for graphs with independence ≤ 2, 399: 407

399. *range of positive eigenvalues* $\leq$ *matching.*

400. *range of positive eigenvalues* $\leq$ *maximum of Dual Degree.*

401. *minimum of derivative of positive eigenvalues* $\leq$ *mean Gravity. Disproved by* **Tony L. Brewster, Michael J. Dinneen and Vance Faber** *12. 90.*

402. *n / mean distance* $\leq$ *largest eigenvalue of Laplacian.*

403. *n / mean distance* $\leq$ *scope of eigenvalues. Disproved by* **Favaron, Maheo and Sacle**. *12. 89.*

404. *2-nd largest eigenvalue of Distance* $\leq$ *number of triangles.*

404' *2-nd largest eigenvalue of Distance* $\leq$ *size / mean Gravity.*

405. *- smallest eigenvalue of Distance* $\leq$ *matching of G + the matching number of the complement of G. Disproved by* **Michael J. Dinneen, Los Alamos National Laboratory and University of Victoria, Victoria, B.C.** *August 91.*

406. *maximum of Even* $\leq$ *chromatic number.* **Favaron, Maheo and Sacle** *disproved this conjecture, but they proved that maximum of Even* $\leq$ *chromatic number +1. [FMS1]. October 88.*

407. *mean of Even* $\leq$ *n - matching.* [**FMS1**]. *November 88. They proved that if the independence* $\leq$ *2 then mean of Even* $\leq$ *n/2 with exception of two one-integer-parameter families of counterexamples. It is interesting to compare this theorem with Shearer's conjecture in 111. Note that the independence* $\leq$ *2, iff the complement is triangle-free. Let e and e' denote respectively mean of Even in G and the complement of G. What is f(n) = minimum(e(G)+e'(G)) where G is triangle-free graph with n vertices.*

422. *If G is a regular graph then n - sum of positive eigenvalues* $\leq$ *independence.*

October 15. 88.

434. *Let S be a set of integers, and let RP[S] be the graph whose vertices are elements of S, two being adjacent iff they are relatively prime, and let PR[S] denotes the complement of this graph. Laplacian of a graph G is the matrix indexed by vertices of G, the diagonal entries are the degrees of the corresponding vertices. Off the diagonal the entries are -1 if the corresponding vertices are joined by an edge, and 0 - otherwise.*

Conjecture: If S is $\{2..r\}$, where r $\geq$2 is a prime then the second largest eigenvalue of the Laplacian of PR[S] is equal to r-1.

Graffiti also made a conjecture that the largest eigenvalue of Laplacian is equal to r-1. These conjectures are correct and they follow from these three facts:

1. Bertrand postulate, which states that for every k, there is prime between k and 2k.

2. Let p be a prime $\geq$ r/2 and R the corresponding row of the characteristic equation of the Laplacian of PR[S]. If the value of the variable is r-1 then every entry of R is equal to -1.

3. The sum of every row of the matrix from 2 is 1 - r.

Now, if r = 2k +1 then there is a prime between k and 2k, and the other prime is r itself.

In particular the multiplicity of r-1 as an eigenvalue is at least the number of primes between r/2 and r. I do not remember testing if they are equal even for small r, but I think they should be close, comp. Ramsey letter in 446. 9.88.

446. *the number of non-negative eigenvalues of* $PR[2,..r]$ *is equal to* $\pi(r)$ *- the number of primes smaller or equal to r. (434.)*

comp conj 434, 470 and 800:813.

In on direction the inequality follows from Cvetkovic Theorem: $\pi(r)$ *is the independence number of* $PR[2..r]$*, which by his result is not more than the number of nonnegative eigenvalues. s.f. 10. 88.*

Below are slightly modified excerpts from a letter of **Keith Ramsay, University of California at Berkeley***, in which he describes a counterexample, barring numerical errors, which seem to me unlikely. Ramsay discusses the problem in terms of the matrix A + I, where A is the adjacency matrix of PR[2..n] April 94:*

"The adjacency matrix of the graphs is decomposable as the product BDB^t*, where* B *is the* n *by* n *matrix where* $B(i,j) = 1$ *if* j *divides* i*, and 0 otherwise, and* D *is the diagonal matrix with* $D(i,i) = -\mu(i)$ *for* $i \geq$ *0, and 0 otherwise, and* t *means transpose. This can be shown by the "mobius inversion formula";* $\mu(i)$ *is the mobius function which is* $(-1)^k$ *when* i *is the product of* k *distinct primes and is 0 when the prime factorization of* i *includes powers higher than 1 (i.e., when* i *is not "square free").*

The matrix B is lower triangular, D is diagonal.

Until you get to n=30, all of the numbers for which mu(i)=-1 are primes. Then $\mu(30)$*=-1. I tried calculating the 31×31 case, and it had the "right" number of eigenvalues* ≥ 1*, but there was an eigenvalue between 0 and 1. I tried n=50, and the number still was right, but there were two eigenvalues between 0 and 1 (corresponding to 2×3×5 and 2×3×7 perhaps?).*

(this intrigues me for a long time: unlike it is the case with eigenvectors, whose components match rows of its matrix, there is no such "natural correspondence" between eigenvalues and rows. Nevertheless in some special situations as above, one is tempted to link eigenvalues with rows. Is there a special class of matrices for which this correspondence can be sensibly defined? s.f)

The image of the matrix is the set of vectors $(a_1, ..., a_n)$ *which have the property that if i and j have the same prime factors (not counting multiplicity) then* $a_i = a_j$*; I can show this with a little manipulation. Thus the rank of the matrix is the number of square-free numbers* $\leq n$*. You can divide up the coordinates into groups according to what primes divide the index. (comp. conj. 470, s.f)*

Then you can, if you like, write a new matrix with rows and columns indexed by square-free numbers, which will have the same eigenvalues. The (i,j) entry is 0 if i and j have no common factors, and is the number of integers between 1 and n which have the prime factors of j as prime factors, if i and j have a common factor.

For smallish n, the number of integers which have 2,3,5 as prime factors (and nothing else) is small compared to the number of numbers which are products of just two out of the three. Eventually however the numbers of the form $2^i \times 3^j \times 5^k$ *where i,j,k* ≥ 0 *outnumber those of the forms* $2^i \times 3^j$ *etc. Thus I think there is a possibility that your pattern will break down once you get to large enough n, by e.g. getting an eigenvalue related to 30 which is greater or equal to 1.*

I notice that some of the eigenvalues appear to be close to the golden mean (1+sqrt(5))/2 and (1-sqrt(5))/2 (perhaps they are equal; Maple could be giving me round-off error).

Just now I had Maple calculate the eigenvalues for the matrix in the n=90 case. It appears that there are 25 of the requisite eigenvalues, but only 24 primes $\leq$ *90. I wouldn't take this as proof, since I don't know how accurate Maple is at calculating eigenvalues of 90*$\times$*90 matrices, but it leads me to expect that your conjecture breaks down by n=90.*

It seems all the 1 eigenvalues correspond to primes are $\geq$ *n/2 and* $\leq$ *n.*

Keith Ramsay,"

A comment concerning the last sentence:

The number of primes $\geq n/2$ *and* $\leq n$ *is smaller or equal to the number of eigenvalues = 1. In the terms of the adjacency matrix, these are 0 eigenvalues and they are induced by primes between* $n/2$ *and* n *which are isolated vertices of* $PR[2..r]$

If indeed we had the equality than it would as interesting as original conjecture, but this seems to me now unlikely. This conjecture is probably closely related to my question from 434.

It would be equally interesting to know whether asymptotically the original conjecture as well as the last statement of Ramsay's letter are the same as the number of corresponding eigenvalues. Concerning the eigenvalues related to the golden ratio, I don't think they are round-off errors. Let p and q be two primes and consider the graph H = PR[2,p,q,2p,2q]. Let g be the golden ratio. Then -g is an eigenvalue of G, and has eigenvector E= (0,-1,1,g,-g). E can be often extended from G to an eigenvector of PR[S] if this graph contains H. For example, if S consist of 2, a set primes and their duplicates then -g (and g-1) is a multiple eigenvalue. s.f 4.94.

Which subgraphs (or principal submatrices) have the property that an eigenvector can be extended to the whole graph with the same eigenvalue? 2.96.

The main (original) motivation of the Riemann Hypothesis was (9. 97 and I guess still is 9. 97) to find the difference between $\pi(n)$ *and the Prime Number Theorem and other estimates of* $\pi(n)$*. RH implies that*

$$\pi(n) = \int_2^n \frac{dt}{logt} + O(n^{0.5}logn).$$

I asked Andrew Odlyzko, ATT, and he told me what I hoped for namely that to prove RH it is enough to find an estimate of $\pi(n)$ *which is at least as good as Riemann's. Since the original conjecture is true in one direction it is quite a* *motivation to estimate p^+, i.e, the number of nonnegative eigenvalues of PR[2..n]. 4. 95.*

Douglas West, University of Chicago, *observed that $\pi(n)$ is bounded below, by the intersection number of PR[2..n], i.e, the minimum number of elements of a set on which the graph can be represented as an intersection graph. By Erdos-Goodman-Posa Theorem this number is the minimum number of cliques covering all edges. Note that in related 470, graphs are naturally represented as intersection graphs on primes from S. 4.95.*

I observed thus, that Graffiti was not the first to make contributions to graph theory, without knowing anything about the subject. It was my advisor Edward Marczewski, or at least that is what he told me. Marczewski proved that every graph is an intersection graph. He also proved another simple result which by now is in sophomore textbooks (and in my opinion it is difficult to do much better than that,) namely that every poset can be extended to a chain. Marczewski wrote other papers which had impact well out of scope of their difficulty, [M].

The above remarks may be good motivation to find out more about eigenvalues of intersection and more generally of the following graphs: for a poset P let G(P) be the graph whose vertices are elements of P, two being adjacent iff the principal ideals generated by these elements intersect.

The definition of an eigenvector of G(P) clearly indicates the relation to the inclusion-exclusion principle (perhaps for the same reason as in Ramsay's letter). Perhaps the general theory of Mobious function can be used to estimate p^+ for arbitrary posets P. RH is equivalent to the statement that for every ϵ sum $\mu(k) = O(n^{1/2+\epsilon})$, where μ is the classical Mobious function, and the summation is over all k smaller or equal to n.

The role of primes, in the case of arbitrary posets will be assumed, by maximal filters, and perhaps one make use of Priestley's or other dualities for posets to get an additional insight into the problem of how large can be the difference between the number of nonnegative eigenvalues of G(P) and the number of elements of P - the dual of P. The number of nonpositive eigenvalues is also an upper bound for the independence, and hence* $\pi(n)$, *but usually the number of negative eigenvalues is larger than the number of positive, which is probably why the program did not make the other conjecture. There are many graphs for which the independence number is equal to the number or nonpositive or nonnegative eigenvalues and they are called here plants (see 345.) Conjectures about plants may be an alternative way to understand why in PR graphs the independence number is close to* p^+. *s.f. 5. 95.*

comp 814.

Bill Staton *proved that every graph is an induced subgraph of PR[n]. Actually at the time we discussed closely related 448, but his comments seem to belong rather here:*

My proof was like this: Make an induction assumption. Delete vertex v *and get a representation of* $G-\{v\}$. *Suppose neighbors of* v *are* $w_1..w_k$. *Find primes* $p_1...p_k$ *relatively prime to all vertices of* $G-\{v\}$. *Alter the representation of* $G-\{v\}$ *by replacing* w_i *with* $p_i \times w_i$ *(this affects no adjacencies in in* $G-\{v\}$*). Now let* v *be represented by the product* $p_1 \times ... \times p_k$.

Bill Staton. Feb 96.

His result seems to me a nice sharpening of Marczewski's Theorem, though as we have noticed, it is easy to deduce it from the latter, by representing G as the set of subsets of the set of primes. Bill also remarks that it may be of interest to find *what is the smallest number $n = f(G)$ such that G is an induced subgraph of $PR[n]$. For example $f(K_n)$ is $2n$. 2.96.*

Bill's theorem is equivalent to infinitude of primes; he obviously used the result, and the converse follows, since embedding of the empty graph E_n requires n primes. The problem of finding $f(G)$ is thus generalization of the problem of distribution of primes, which corresponds to the case of empty graphs. I think that of particular interest is what is f(random graph) and f(Paley graph). Is there any simple sequence of graphs G_n, of bounded degree, or with the independence number tending to infinity, for which there is a formula for $f(G_n)$, or even an asymptotic formula with an error smaller than the Prime Number Theorem gives for empty graphs? s.f. 2. 96.

What is the complexity of the decision problem:

Instance: G - graph, k - integer.

Question: Is $f(G) \leq k$?

I know that Posa had some results or problems, concerning complexity of a graph, i.e, the shortest description of G. Did he write anything about it? The problem is certainly not unrelated to Erdos-Goodman-Posa theorem and now Staton's result, which has the additional arithmetical and perhaps even practical appeal, because of coding problems.

Let us represent edges of G as a minimum number of edge disjoint paths, code vertices as integers and then list them in order in which they appear in paths using $\bullet$ to mark a jump to a new path. For Eulerian graphs this is in a sense a shortest description of a graph, because it requires the space logn $\times$ (the number of edges). Are there results or even fairly well stated questions concerning efficiency of coding algorithms using some extra structures like for example the above Staton's representation.) For some (sequences of) objects the most efficient codes may be obtained *by first describing a coding scheme and then including it as a part of the code. For example for Paley(zillionth prime) graph the most efficient code probably must contain the description of modular arithmetic or an equivalent coding scheme. Perhaps, this could be strictly proved by showing that one can define modular arithmetic in terms of graph-theoretical concepts of quadratic residue graphs.*

2. 98. This is a possibility of trying to resolve (I think) Jules Richard's paradox, which I think goes something like this: Let S be lex-first sentence requiring zillion letters to describe it. 2. 98

Bill Staton told me that he thinks that maximum of f(G) over all n-vertex graphs is the product of n-1 first primes, which would be the best possible because of stars.

The problem what is $f(G_n)$ seems to lead to interesting questions for many sequences of graphs. For example for the complete bipartite graphs $K_{n,n}$ it seems to be the following problem: for a $n \times n$ matrix A, let mx_A be the maximum of p_l - the product of all entries from a line l, where line is a row or a column. What is the minimum of mx_A over all permutations of entries of A. Taking the entries to be the first n^2 primes we get that the minimum is $\geq f(K_{n,n)})$. Conversely, every minimal embedding of $K_{n,n}$ into $PR[2..m]$ defines two orthogonal partitions of primes, and the minimality implies that each part has exactly n elements. 2. 96.

[M], E. Marczewski,

Collected Mathematical Papers, Polish Academy of Sciences, 1996.

447. *The sum of reciprocals of rainbow of PR[2..n] is not more than its independence number. (247.)*

448. *If G is the graph whose vertices are numbers 2..n, two being adjacent iff they are not relatively prime, then the residue of G is at least $\sqrt{n}$.*

Favaron, Maheo and Sacle showed that this conjecture follows from known results on distribution of primes.

Professor **Erdos** *asks if there is a constant* $c \geq 0$ *such that residue* $\leq$ *(1-c) n/logn. He remarks that (0.5+c)n/logn* $\leq$ *residue follows from known results about distribution of primes in the interval [n/2 n]. (see also [FMS1].) Nov. 89.*

Conjecture residue = (2/3 +o(1))n/logn. s.f Nov. 89.

Favaron, Maheo and Sacle and Charles Delorme wrote a program to test this conjecture up to n = 10000 and the and the values seem to be close. They also computed sum $1/(d_i + 1)$ *where* d_i *is the degree sequence of G, and this value is also close to the residue. The result of Favaron, Maheo and Sacle (conj 69) seems to me so nontrivial, that perhaps it is of interest to have a direct proof in the case of graph G. Dec 89.*

(*) Is it possible to find a lower bound for the independence number in graphs which provide an information about distribution of primes? Dec 89.

Paul Erdos and Bill Staton *proved that the residue is at least*

$$(\pi^2/6 - 1)n/logn.$$

The proof is based on the Prime Number Theorem and the fact that if m is prime in the interval (n/(k+1), n/k) then m has degree k in G. A set of k+1 vertices of degree k (in the worst case) leads in the Havel-Hakimi reduction to a set of k vertices of degree k-1, and eventually to the same number of zeros. Since there are about $\frac{1}{(k^2+k)}\ \frac{n}{logn}$ *primes between n/(k+1) and n/k, the counting gives the result. February 95.*

Bill Staton *thinks that the residue of PR(n) is asymptotically* $(\pi^2/6 - 1)$ *n/log(n). He gives the following informal argument: The non primes can't contribute much. Each non prime has a prime factor no bigger than* $\sqrt{n}$ *and is therefore* *in a clique of size at least* $\sqrt{n}$. *These vertices are covered by no more than* $\frac{\sqrt{n}}{log\sqrt{n}}$ *cliques and hence there is no large independent set in the subgraph they induce. That should mean they can't contribute much to the residue of G. 1.96.*

Independently **Professor Erdos**, *asked me to compute the number of Havel-Hakimi reductions needed to obtain the first nontrivial zeros contributing to the residue. The trivial zeros are those resulting from primes greater than n/2 (all roads lead to Rome, and all nontrivial zeros to the Riemann Hypothesis.) I did this for several values up to 200 and the results seem to me very surprising. It looks that with exception of the first few n's, after the appearance of the first nontrivial zero, all terms of the derived sequence are smaller or equal to 1 and this occurs about n(1 - 1/logn) iterations. This seem to corroborate Bill's conjecture about composites making insignificant contributions to the residue. 1. 96.*

Let us call a class of graphs R residual if the sequence derived from the degree sequence can be realized as a degree sequence of a graph from R. Perhaps the problem can be understood better by finding a residual class of graphs containing all PR[n]'s. 2.96.

Every graph is an induced subgraph of PR[n] for some n. **Bill Staton.** *2. 96. There is much more about his simple and nice result in 446.*

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450. *If* G *is the* $RG(n)$*-graph then the matching* $= n -$ *independence.*

* * *

454. *The number of primes not more than n is not more than the number of distinct eigenvalues of RP[2..n] (defined in 434).*

456. *The residue of the graph RP[2..n] is not more than* $\pi(n)$.

RP is defined in 434 and the residue in 69.

Is residue of H bounded by a constant? Paul Erdos and sf.

Professor Erdos thinks that the residue of this graph is even more interesting than that from 448.

457. *The number of quadratic residues mod n is not more than the rank of RP[2..n].*

Two vertices in RP are adjacent iff they are relatively prime.

458. *Let e(v) be the number of vertices at even distance from v in RP[2..n] defined above. The sum of reciprocals of e(v) is not more than* $\pi(n)$.

470. *Let S be a set of integers, and let PR[S] be the graph whose vertices are elements of S, two being adjacent iff they are not relatively prime. Conjecture: If S is the set square-free integers from the interval [2,..r] then the number of nonnegative eigenvalues of PR[S] is equal to p(r) - the number of primes smaller or equal to r.*

comp conj 446.

Perhaps this question is of interest: Let us call a subset S of integers greater than 1, ***spectral*** *if*

1. S contains all primes. and

2. The number of nonnegative eigenvalues of PR[S(n)] is equal to the number of primes not greater than n, where S(n) is the set elements of $S \leq n$ *Clearly the set of all primes is spectral and the union of increasing sequence of spectral sets is spectral. By Ramsay's letter from 446, the set of all integers more than 1 is not spectral.*

Can one give an explicit example of a maximal spectral set? Is there a spectral set of positive density? s.f. 4.94.

It looks that the first integer for which the number of primes from S is different from the number of nonnegative eigenvalues of PR[S] is 210, and the difference does not increase at least up to 255. I again used the Eispack. s.f 5.95.

December 18, 88.

Conjectures 494 - 536 are about Paley graphs.

S is the set of quadratic residues mod n, it is treated here also as vector whose components are elements of S .

495. *length of coordinates of Maxine* $\leq$ *2(chromatic number)*

496. *sum of coordinates of Maxine* $\leq$ *length of S.*

497. *size - order* $\leq$ *the number of triangles.*

503. *frequency of minimum of rainbow* $\leq \pi(n)$.

504. *the number of square-free integers not exceeding n and being products of even number of primes ¡ sum of reciprocals of coordinates of Maxine.*

509. *mean rainbow* $\leq$ *frequency of maximum of eigenvalues of Laplacian.*

513. *chromatic number of a Paley graph with n vertices is not more than the number of primes not more than n.*

A recent unpublished yet result of S.Graham and C. Ringrose (see also [CGW]) combined with the Prime-Number Theorem, may be an indication that the two invariants are not far-off. Graham and Ringrose proved that for infinitely many n, the largest independent set in Paley graphs has size $O((logn)(logloglogn)$*, [GR]. From earlier results of Montgomery assuming the Generalized Riemann Hypothesis, we had only a lower bound* $O((logn)(logloglogn)$*, infinitely often. Paley graphs are considered to be deterministic models of random graphs, [BT], [CGW], [GS], and [TH]. Together with the above results, this may be an indication that the conjecture is correct, for infinitely many n, and hopefully even related to RH. September 89.*

515. *pi(n) sum of reciprocals of coordinates of a maximum clique.*

516. *deviation of S $\leq$ frequency of mode of eigenvalues of Laplacian.*

523. *mode of eigenvalues of Laplacian $\leq$ number of square-free integers not greater than n.*

528. *the number of square-free integers not exceeding n and being products of odd number of primes $\leq$ 2(chromatic number).*

529. *deviation of S $\leq$ number of cubic residues less than n.*

536. *The number of cubic nonresidues $\leq$ n / average distance of Paley graph with n vertices.*

The average distance of this graph is 3/2 *and when I told this to* **Andrew Odlyzko***, he said that the conjecture is obvious: the left side is either equal to zero or to the rounded value of the right side. February 89.*

January 11, 89

537. *If G is a connected Cayley graph of cyclic groups with at least two vertices then chromatic number $\leq$ rank.*

538. *If G is a connected Cayley graph of cyclic groups with at least two vertices then the maximum of the rainbow of G is $\leq$ number of negative eigenvalues.*

February 4, 89.

Let D be the (sorted down, i.e, starting with the largest number) degree sequence of graph. Suppose that the first, i.e. largest component of D is p. The derived sequence is obtained from D by deleting this first component and subtracting 1 from the p following components. k-Residue is the vector obtained by repeating this operation until all components are not more than k. The mid-Degree is vector obtained from the degree sequence by repeating this operation depth/2 (rounded up) many times.

543. *$n - independence \leq$ the sum of positive eigenvalues. This conjecture is related to 20 and 21, because of the two known results, which were also conjectures of Graffiti: n - independence number of positive eigenvalues and n - independence number of negative eigenvalues.*

547. $independence \leq n - mode of mid - Degree.$

548. *mode of mid-Degree* $\leq \frac{size}{independence.}$

552. *independence* $\leq$ *n - mean of mid-Degree.*

553. *mean of mid-Degree* $\leq \frac{size}{independence.}$

561. *If G is a connected graph then the mean of Rainbow* $\leq \frac{size}{independence}.$

568. *If G is a connected graph then the number of positive eigenvalues - number of negative eigenvalues* $\leq \frac{size}{independence.}$

574. If G is a connected graph then the $\frac{chromatic number of complement of G}{independence} \leq$ *mode of Even.*

February 9, 89.

Conjectures for trees 577:594

577. *If G is a tree then the radius* $\leq$ *inverse Dual Degree.* **Shi Ronghua, Dept. of Applied Mathematics, East China Institute of Technology, Nanjing** *proved that for every tree 1/3 + [0.5 diameter]* $\leq$ *sum of inverses of dual degree [SR]. September 90.*

Man-Keung Siu, The University of Hong-Kong, *bf Zhongfu Zhang, Lanzhou Railway Institute, PRC and* **Sanming Zhou, The University of Western Australia** *proved in addition that if the diameter is even then the invariant on the left is not less than radius* $+5/6$. *6. 98.*

578. *If G is a tree then the radius* $\leq$ *range of positive eigenvalues. Siemion Fajtlowicz. February 89.*

579. *If G is a tree then the maximum of coordinate of Maxine $\leq$ maximum of Dual Degree.*

582. *If G is a tree then the independence $\leq$ number of componnents of 1-Residue.*

584. *If G is a tree then the maximum eigenvalue of Laplacian $\leq$ 2 + independence.*

Conjectures 595 - 605 are about triangle-free graphs.

* * *

595. *chromatic number of the complement of $G = n$ - the matching number.*

This is true, and in particular there is a polynomial-time algorithm for chromatic number for graphs with independence 2. I am fairly sure that, this is not the case, for graphs with independence = 3. For every graph we have that chromatic number its complement complement $\leq$ n - the matching number and and for more than half in the database of Graffiti we have the equality. Most of them are however triangle-free. (comp. Molloy's results in 822.)

* * *

596. *radius maximal frequency of mid-Degree. Disproved by Tony L. Brewster, Michael J. Dinneen, and Vance Faber, August 90.*

597. *radius $\leq$ maximal frequency of Even.*

598. *range of coordinates of matching $\leq \frac{n}{averagedistance.}$ Michael J. Dinneen, Los Alamos National Laboratory and University of Victoria, Victoria, B.C (comp. 107.) August 91.*

599. *n - independence $\leq$ chromatic number of G + the chromatic number of complement of G.*

Favaron, Maheo and Sacle used 595, and proved another lemma of their own to disprove this conjecture, FMS 10, 89.

[FMS3], Favaron, Maheo and Sacle, On Conjectures of Graffiti, III.

600. *mean of rainbow* $\leq$ *inverse Odd.*

601. *chromatic number* $\leq$ *n / average distance. disproved by Peter Puget, June 90.*

602. *n / independence* $\leq$ *range of coordinates of Maxine.*

603. *mean of dual degree* $\leq$ *mean of Even.*

604. *mean of Even* $\leq$ *chromatic number + chromatic number of the complement.*

605. *maximum of Odd* $\leq$ *chromatic number + chromatic number of the complement of G.*

607. *mean Rainbow* $\leq \frac{size}{independence.}$

February 12, 89.

Conjectures 634 - 654 are for graphs in which chromatic number of complement of G = n - matching.

According to 595, every triangle-free graph has this property which is my motivation for including these conjectures. Compare for example 70 and 640; conjecture 70 is true for triangle-free graphs. Also every graph in which n = matching + independence has the property in question.

634. *inverse of coordinates of Maxine* $\leq$ *n /2. comp 212. FMS.10, 89.*

635. *size/independence* $\leq$ *chromatic number of G +chromatic number of the complement of G.*

636. *size / independence* $\leq$ *maximum eigenvalue of Laplacian.*

637. *size / independence $\leq$ sum of positive eigenvalues. Disproved by Michael J. Dinneen, Los Alamos National Laboratory and University of Victoria, Victoria, B.C (comp. 107.) August 91.*

638. *maximum of Rainbow $\leq$ n /2.*

639. *mean Rainbow $\leq$ Randic.*

640. *chromatic number $\leq$ maximal frequency of coordinates of maximum clique. comp. 70.*

641. *chromatic number $\leq$ frequency of maximum of Rainbow.*

642. *scope of Dual Degree $\leq$ independence. FMS, December 89.*

643. *scope of Dual Degree $\leq$ number of components of mid-degree. FMS, December 89.*

644. *minimum of Dual Degree mean of Odd. Michael J. Dinneen, Los Alamos National Laboratory and University of Victoria, Victoria, B.C (comp. 107.) August 91.*

645. *mean of Dual Degree $\leq$ chromatic number + chromatic number of complement of G.*

646. *Randic $\leq$ maximal frequency of coordinates of a maximum clique.*

Michael J. Dinneen, Los Alamos National Laboratory and University of Victoria, B.C *(comp. 107.) August 91.*

* * *

647. *there are graphs with chromatic number of complement of G / independence $\geq$ residue of the complement.*

648. *there are graphs with chromatic number of complement of G/ independence geq average distance.*

In both cases examples are sufficiently large triangle-free Ramsey graphs. By 595 these graphs satisfy chromatic number of complement of G = n - matching. The first conjecure then follows from 69 and the second is true , because we can always assume that the diameter of Ramsey graph is 2. Perhaps there are simpler examples, but Graffiti does not know any counterexamples to its own conjectures; nevertheless it predicts now sometimes that some of them are false.

* * *

649. *there are graphs with chromatic number of complement of G/ independence $\geq$ range of positive eigenvalues. Tony L. Brewster, 6.91.*

650. *maximum eigenvalue $\leq$ chromatic number of G +chromatic number of the complement of G.*

651. *average distance $\leq$ maximal frequency of Degree. Michael J. Dinneen, Los Alamos National Laboratory and University of Victoria, Victoria, B.C (comp. 107.) August 91.*

652. *average distance $\leq$ inverse dual degree. Michael J. Dinneen, Los Alamos National Laboratory and University of Victoria, Victoria, B.C (co 653. average distance frequency of mode of 1-Residue. Michael J. Dinneen, Los Alamos National Laboratory and University of Victoria, Victoria, B.C (comp. 107.) August 91.*

654. *minimum of Even $\leq$ chromatic number of G +chromatic number of the complement of G.*

February 14, 89.

Conjectures for graphs with sum of Even $\leq$ sum of Odd, 655 : 688

656. $\frac{size}{independence} \leq$ *sum of coordinates of a maximum clique.*

657. *mean Rainbow* $\leq \frac{size}{independence.}$

662. *deviation of eigenvalues $\leq$ n - independence.*

m_0 and m_1 denote respectively the multiplicity of 0 and 1 as the eigenvalues over the 2-element field. One can think about eigenvectors over GF(2) as sets of vertices.

693. *independence $\leq n - m_1$.*

694. *there is an eigenvector E belonging to the smallest eigenvalue such that the frequency of maximum of E $\leq$ independence. for regular bipartite graphs we have the equality.*

695. *range of nonpositive eigenvalues $\leq 1 + n - m_0$.*

696. *- (mean of nonpositive eigenvalues) $\leq$ chromatic number of complement of G.*

697. *range of the largest eigenvector $\leq n - m_1$.*

698. *lenght of negative eigenvalues $\leq$ the Randic Index.*

699. *average distance $\leq$ sum of reciprocals of square roots of degrees. comp. [EPS] and comments to 4.*

700. *deviation of distance $\leq$ residue. Peter Puget, June 90.*

N

ovember 11, 89.

Eigenvectors are oriented so that the maximum is nonnegative and the sum of absolute values of the components is n. Unless it is explicitely mentioned eigenvectors mean eigenvectors of the adjacency matrix. They are in general try-outs rather then invariants and perhaps a reasonable interpretation of a conjecture involving eigenvectors is an additional assumption that the eigenvector in question is unique. For example for conjectures involving the largest eigenvector (i.e, belonging to the largest eigenvalue) it might be an assumption that G is connected. Is there a similar result for graphs with unique smallest eigenvalue?

701. *The average distance* $\leq$ *the inverse rainbow.*

702. *The mean temperature* $\leq$ *the mean rainbow.*

704. *The range range of rainbow* $n - m_1$

705. *The diameter* $\leq m_0$.

706. *the matching number* $\leq$ *the sum of positive eigenvalues.*

A stronger result is true: let k be the chromatic number of the complement of G. Then the sum of the first k eigenvalues is at least n-k. Odile Favaron, Maryvonne Maheo and Jean-Francois Sacel. December 89.

707. *The radius* $\leq$ *number of positive components of the smallest eigenvector.*

708. *let* V *be the vector of positive components of the smallest eigenvector and let* v *be the scalar product of* V *with itself. Then the average distance* $\leq v$.

709. *the maximum of the largest eigenvector* $\leq$ *the residue.*

710. $m_0 \leq n-$ *the residue.*

711. *The range of deficiency* $\leq$ *the range of eigenvalues.Tony L. Brewster, Michael J. Dinneen and Vance Faber, (see107) 12. 90.*

712. *The minimum temperature* $\leq$ *number of nonpositive eigenvalues.*

713. *- mean of nonpositive eigenvalues* $\leq$ *Randic.Odile Favaron, Maryvonne Maheo and Jean-Francois Sale. 2. 90.*

714. *- mean of nonpositive eigenvalues* $\leq$ *sum of reciprocals of all temperatures.*

715. *the scope of nonpositive eigenvalues* $\leq$ *mean of degrees greater or equal to the average degree.Tony L. Brewster, Michael J. Dinneen and Vance Faber, (see107) 12. 90.*

$* * *$

3. 90.

716. *the diameter - radius $\leq$ the matching number.*

Which graphs have a spanning trees with the same radius and (or) maximum matching as the graph itself?

717. *The mean degree $\leq$ mean dual degree,* **Vance Faber**, *Los Alamos National Laboratory, June 90.*

7. 98. This conjecture was also proved (and discovered) by **FMS** *independentely from Graffiti, though in their paper on conjectures of the program published in the Proceedings of the 12th BBC (Norwich 1989), Ars Combinatoria 29 C, 90-106, 1990 (see Section 2.2, Proposition 1 after Conjecture 245). 7. 98.*

It would be interesting to know how much can the mean of dual degree deviate from the mean degree. For example, is it true that the maximum difference for a fixed number of vertices occurs in stars? A reason why the difference is of interest is the following statistical interpretation: suppose that a sociologist wants to measure an average level of friendship in a group, but she is concerned that if she'll ask each person how many friends he has, the answer may be biased. Instead she asks each person how many friends on average his friends have. Averaging answers seem to be at first sight an unbiased (also in statistical sense) estimator of the mean. However, we will get the correct estimate only if every person has, essentially, the same number of friends; Faber's proof implies that equality holds true iff every component of graph is regular. A somewhat similar, but it seems that a simpler situation is discussed in [HM]. Faber generalized his proof to nonnegative matrices. 7.90.

John Burghduff run the program to generate conjectures about the difference of these two invariants and he proved the first of the two below namely that

* * *

718. *mean of dual degree - mean degree $\leq$ scope of degree. 8.90. This is a good bound in the sense that both sides of the inequality are very close in stars. Among other conjectures found by him was*

* * *

719. *mean of dual degree - mean degree $\leq$ scope of dual degree.Tony L. Brewster, Michael J. Dinneen and Vance Faber, (see107) 12. 90.*

Burghduff also proved the following conjecture of Graffiti and its dual version for geotropic plants:

720. *For every heliotropic plant, Randic is not more than the number of non-positive eigenvalues.*

Looking at his proof we have noticed that the conjecture follows from the following statement: For every graph the independence number is not more than 3n/2 - rank. Even though this last statement can be easily deduced from Cvetkovic's spectral bound for the independence number (see the definition of plants) I consider it interesting because it seems hardly obvious for which graphs we have the equality and it seems that there are a lot of plants with this seemingly very strong property. Graphs for which the independence number = 3n/2-rank will be called here perfect plants.

Actually Cvetkovic Theorem has a stronger corollary namely that for every graph independence $\leq$ n - rank/2. The proof automatically implies that graphs satisfying independence = n - rank/2 must have the same number of positive and negative eigenvalues and thus that they are both helio and geotropic plants . I found a direct proof of independence $\leq n - rank/2$. 1. 91.

John Burghduff proved that if G is a perfect plant then matching + independence = n. This generalizes his result in 721. 1.91.

721. *If G is a perfect plant then its rank is equal to the number of vertices. Burghduff also proved that every perfect plant has a perfect matching. This seems to indicate that indeed there are plenty of perfect plants. Let us consider a matching M such that every edge of M joins a red and a blue vertex. Let G be random graph containing M and subject to the condition that blue vertices are independent. I would conjecture that with probability 1, G must be an perfect plant. 9.90. Let G be a graph, let I be a maximal independent set of G and E its complement. Let* G^* *be any graph obtained from G by deleting any set of edges joing vertices of E. If G is a perfect plant then* G^* *is a perfect plant and I is a maximum independent set in* G^* *. Conversely if I is a maximum independent set in a perfect bipartite plant B and G is obtained from B by adding any set of edges in E then G is also a perfect plant. This of course proves the above conjecture. S. F. 1.90.*

* * *

August 19. 90

The next four conjectures were made by Graffiti instructed by Burghduff and myself to search for conjectures related to Cvetkovic's spectral bound for the independence (comp definitions of plants.) They were selected out of 9 conjectures, seven of which were tested in Los Alamos, against all graphs with at most 10 vertices. Six of the seven tested in Los Alamos were shown to be false. (see conj. 107.) Conjecture 723 below passed this test.

Conjectures for all graphs

722. *the number of nonpositive eigenvalues - frequency of mode of the eigenvector of the largest eigenvalue* $\leq$ *independence .*

723. *the number of nonnegative eigenvalues - sum of reciprocals of eigenvalues of Laplacian* $\leq$ *independence.*

724. *the number of nonnegative eigenvalues - largest eigenvalue + smallest nonnegative eigenvalue* $\leq$ *independence. Tony L. Brewster, Michael J. Dinneen and Vance Faber, (comp 107) February 91. The only counterexample they found in their search was a disjoint union of two* C_5s.

725. *the number of nonnegative eigenvalues - sum of reciprocals of coordinates of Maxine* $\leq$ *independence .*

Almost all new programs of Graffiti including the Geometry, whose con- jectures are listed below, were written jointly with my student Ermelinda DeLaVina. The development of this version of the program was supported partially by grant 003652085-ARP.

Summer 90

I use here these definitions : a ***polygon*** *is a closed, piecewise-linear curve. If the curve does not intersect itself and contains at least three vertices then the polygon is called* ***simple****. The straight line segments of a polygon are called its sides, and their endpoints are called vertices of the polygon. The distance from point p to side s is the distance from p to the line containing s. We shall assume that all polygons are non-degenerated i.e they have no zero sides nor angles equal to zero. Two points are said to be* ***visible*** *with respect to a polygon if the segment joining these two points does not intersect other sides of the polygon. In particular, unless there is an "obstruction," two points along a side are visible. The graph whose vertices are points of the polygon, two being adjacent iff they are visible is called the visibility graph of the polygon. Conjectures below are about finite* ***configurations*** *of points on the plane. A configuration can contain repeated points, as it for example, may be the case with a (non simple) polygon. n will always denote the order of the configuration.*

* * *

726. *every simple polygon contains three mutually visible vertices.*

This was the very first conjecture made by the geometric version of Graffiti.

* * *

Among the first dozen or so conjectures there were the next two which with analogous formulation are valid for distance matrices of trees, [GP]. I think it would be interesting to know, for what metric spaces are they true, in general.

727. *The distance matrix of p distinct points on the plane, $p > 1$, has exactly one positive eigenvalue. This conjecture is true, see 728.*

728. *The distance matrix of p distinct points on the plane, $p > 1$, has no zero eigenvalues.*

Shunshua Sun, Sechuan University, *outlined to* **Vern Paulsen, UH** *an argument that if this conjecture is true for points on a line then it is true for the plane. Paulsen later told me that he proved the conjecture for the line. Summer 90.*

James B. Shearer *showed that 727 implies 728. He thinks that these conjectures are equivalent and that both are true. Summer 91.*

Brad Baxter, Cambridge University, told me that the conjecture is correct and was proved by Schoenberg, [S]. March 94. Below is a quote from his letter:

This remarkable result arose in the study of isometric embeddings = into Hilbert space. In fact, Schoenberg's proof shows that

sum y(j) y(k) A(j,k) ≤ 0

for any real numbers y(1), ..., y(n), not all zero, whose sum is zero. Since the trace of A is zero, there are n-1 negative eigenvalues and one positive eigenvalue. The result is part of a larger corpus of similar results used in the theory of radial basis functions. See for example, Micchelli (1986) or Baxter (1991). It also crops up *in the theory of convex bodies and the moment problem (see Akhiezer, The Classical Moment Problem").*

References:

B. J. C. Baxter (1991), "Conditionally positive functions and p-norm= distance matrices", Constr. Approx. 7, 427–440.

C. A. Micchelli (1986), "Interpolation of scattered data: distance matrices and conditionally positive functions", Constr. Approx. 2, 11= –22.=20

[S] I. J. Schoenberg, On certain metric spaces arising from Euclidean space by a change of metric and their embedding in Hilbert space, Ann. of Math. 38, 787–793.

729. *For every configuration the average transmission of a point is not more than the largest eigenvalue of its distance matrix.*

The ***transmission*** *of a point is the sum of distances from this point.*

This is true because the largest eigenvalue of a nonnegative matrix is at least as large as the average row sum of the matrix. It would be interesting to know for what sets of points the equality holds true. In particular we have this question : suppose that all elements of S have the same transmission. Does the group of isometries of S acts transitively on S.

Working on a related conjecture of Graffiti, which turned to be false because of a numerical error, I noticed that the characteristic polynomial of the distance matrix of a triangle has the form $x^3 + px + q$, which happens to be the reduced form of the general cubic equation. Since the formula for roots of this equation involves trigonometric functions (in the case of three real roots) I wonder if there is a geometric interpretation of roots in terms of a suitable triangle.

730. *If P is a polygon without multiple points then its minimum angle is not more than than the mean degree of its visibility graph.*

Is it posible to construct a polygon whose visibility graph has no edges at all?

Every polygon contains at least three pairs of visible vertices. What is the lower bound for the number of such pairs, for example as a function of number of vertices?
Paul Erdos *2. 91.*

731. *Minimum angle of a polygon without multiple vertices is not more than the minimum degree of the complement of its colinearity graph.*

The vertices of colinearity graph of a configuration are points of the configuration, two being adjacent iff the straight line they determine, contains a third point of the set. If correct this conjecture would generalize the Gallai-Sylvester theorem.

732. *Let L be the sum of reciprocals of nonzero degrees of the visibility graph of a polygon, and let R be the number of distinct components of the eigenvector corresponding to the smallest eigenvalue of its distance matrix. Then $L \leq R$. R is not uniquely defined and hence it is not an invariant, but what I call a try-out, and as such it admits several interpretations. I personally believe that the stronger the conjecture the better, even though in this case the conjecture is more likely to be false. In a sense I consider all nontrivially false conjectures interesting, because proofs that they are false usually, though not always, provides us with new examples. Hence I believe, that the best interpretation of a conjecture involving a try-out, is the strongest one i.e, such that the conjecture is valid for all possible interpretation of the try-out. Another reasonable interpretation is to limit the strongest version of a conjecture to a situation in which the try-out is uniquely defined. In this case it might be an assumption that the distance matrix has a unique smallest eigenvalue; many polygons and graphs in Graffiti's database have this property. Actually the new version of Graffiti defines properties to make conditional conjectures. In the past to get conditional conjectures, properties had to be defined by a user. One of* *the first properties defined by Graffiti was class of graphs with the unique smallest eigenvalue, which seems to me very interesting concept, which as far as I know was not studied at all. Let c be the smallest constant such that L cn? It is easy to see that for simple polygons, c $1/2$, but I would think that c is strictly smaller than $1/3$.*

733. *The number of distinct degrees of the interval graph of a polygon is not more than the number of vertices of the convex hull of the polygon.*

Definition. The vertices of the interval graph of a configuration are straight-line intervals whose endpoints are coordinates of points of the configuration. Two intervals are defined to be adjacent if they have nonempty intersection.

734. *The sum of reciprocals of nonzero degrees of the colinearity graph of a polygon is not more than the chromatic number of its visibility graph.*

735. *The sum of reciprocals of nonzero degrees of colinearity graph is not more than the number of distinct eigenvalues of its distance matrix.*

736. *For every polygon the minimum degree of its visibility graph is not more than the chromatic number of its visibility graph.*

Programs for some of the elementary geometry invariants, including the three below were written by **Michael Grenado, University of Houston.**

Definitions. The point of intersection of three side bisectors of a triangle T is called the center of the triangle. The point of intersection of three angle bisectors of a triangle T is called the incenter of the triangle. The point of intersection of three altitudes of a triangle T is called the orthocenter of the triangle.

* * *

737. *The three triangles obtained by joining the center to vertices of a triangle have the same area.*

The same holds true for six triangles determined by the three side bisectors.

738. *Let a and s be lengths of angle and side bisectors drawn from the same vertex of a triangle. Then* $a \leq s$.

739. *Let p be an intersection point of two angle bisectors of a triangle and d the distance from p to the side from whose vertices these bisctors are drawn. If p is the perimeter of the triangle then the area of the triangle is dp/2.*

* * *

Erdos conjectured in '35 that for every point p inside of a triangle T the ratio of the sum of distances from p to vertices T and the sum of distances from p to sides of T is always ≥ 2. *This was proved two years later by Mordell. The 3-dimensional generalization is still an open problem. Let p be a point inside of a polygon P and let r(p) be the ratio of the sum of distances from p to vertices of P and the sum of distances from p to sides. By a distance to the side I mean the distance to the straight line determined by this side. The point realizing the minimum of r(p) will be called the Erdos-Mordell point of the polygon and the value of r at this point will be called the Erds-Mordell value of the polygon. The location of the Erdos-Mordell point is an open problem even in the case of triangles, but examples and conjectures seem to indicate that the point often tends to lean toward the vertex which determines the smallest angle. I will refer to this vertex as the smallest vertex and to two others respectively as the largest and the median vertices. Apart from simply scanning the points of the triangle, I do not know any algorithm for computing the coordinates of EM point. The problem seem to be of interest for its own sake, but in the case of Graffiti it might help to increase the speed of computation and correctness of conjectures.; I believe that uniqueness of EM point is also an open question.*

741. *The distance from Erdos-Mordell point to the smallest vertex is not more than the sum of distances from the orthocenter of a triangle to its sides.*

742. *The distance from center of the triangle to the smallest vertex is not more than the maximum distance from Erdos-Mordell point to vertices.*

743. *The distance from center of the triangle to the largest vertex is not more than the distance from Erdos-Mordell point to the largest vertex.*

* * *

An experiment in implementing quantifiers produced

744a. *For every two points p and q inside of a triangle, the distance from p to sides is not more than the distance from q to vertices.*

* * *

Summer 92.

747. *Let b be the order of a largest bipartite subgraph of a connected graph G. Then the average distance between distinct vertices of G is not more than $b/2$.*

If correct this conjecture would generalize conj. 2 that the average distance is not more than the independence number. This conjecture was proved by **Fan Chung.**

748. *Let G be a connected graph and let $p(v)$ be the length of the longest path of G starting at the vertex v. Let p be the minimum of $p(v)$ (over all vertices of G. Then the chromatic number of G is not more than $1 + p$.*

The conjecture is correct and it generalizes the well-known result of Gallai. The proposition proved in 756 shows that if G is connected then p is not less than the global minimum degree of G. Hence 748 follows, because the chromatic number is not more $1 + g$.

One can ask various questions of this type: Let k be the chromatic number of G and suppose that G is properly colored with k colors. Let v be a vertex of G. Is there a path starting at v which represents all k colors. s.f. January, 93.

Henk Jan Veldman, University of Twente, Netherlands, *found a different proof of this conjecture. February 93.*

Hao Li, Universite de Paris-sud *learned about this conjecture, and he formulated its digraph-theoretical version, [LI]. Li also answered positively my question above. 4. 98.*

[LI] Hao Li, A generalization of Gallai-Roy Theorem, preprint 98.

749. *Pierre Hansen and Maolin Zheng defined the following game on a connected graph G. There are two players and each of them selects a vertex of G. The second player must choose a vertex different from the first. Each player wins those vertices of G which closer to their selected vertices. In the case of a tie the vertex is won by both players. The market value of G is the number of vertices which can be won by the first player assuming the best play by the second player.*

Conjecture: the average distance of G is not more than the market value of G.

750. *Let e be an edge and let v be a vertex. e is called a v-horizontal edge if the distance from v to both endpoints is the same. Let $h(v)$ denotes the number of v-horizontal edges, and let $d(v)$ be the the number of of vertices at odd distance from v.*

Graffiti made the conjecture that for every connected graph G the independence number of G is not less than max $d(v)$ - min $h(v)$.

I am fairly sure that this conjecture is false, but I do not know any counterexamples. It is not difficult to prove that the independence number is not less than *max(d(v) - h(v)), and there is a similar bound with d(v) replaced by e(v) - the number of vertices at even distance from v.*

In spite of their simplicity both bounds seem to be very good because there are many graphs for which they predict correctly the independence number.

One of **Peter Puget's** *counterexamples (to a previous conjecture) turned to be to be also a counterexample to "independence number is not less than max e(v) - min h(v)," but I do not know examples in which the difference between left and right side of inequality is ¿ 1.*

Let d = max d(v) - min h(v) and let a be the independence number of a graph. **Petr Lisonek, RISC, J. Kepler University, Linz , Austria** *found examples of graphs G in which d is 2a -3. His examples are obtained from bipartite graphs B(n,n) by first subdividing one of the edges and then by replacing it by a complete 4 vertex graph - edge. For a = 4, Lisonek's example was found with a computer, similarly as Puget's which was also a small graph. Nov.92.*

As far as counterexamples are concerned even and odd versions of the conjecture are equivalent because "even" counterexamples can be obtained from "odd" counterexamples G by amalgamating them with the 2-vertex path at a vertex of G realizing the minimum of h. Let p(G) = max(max d(v),e(v)). I think it would be of interest to know for which pairs of integers (k,l) there are graphs G such that k=p(G) - min h(v) and l=the independence number of G. s.f. nov. 92.

Let b = max (max(d(v) - h(v)), max(e(v) - h(v)). Then for every connected graph G the independence number of G is greater or equal to b, and similarly as above there are a lot of graphs for which we have equality, [V].

I have recently received from Gordon Royle a rich library of cubic graphs constructed by a program of Gunnar Brinkmann. The frequency of equality between two *invariants is even greater. April 93.*

Gilles Caporossi, **Pierre Hansen**, *and* **Florian Pujol**,

ISIMA, Clermont-Ferrand, France, *began testing of conjectures involving the independence number. They found with AutoGraphiX a 9-vertex counterexample to this conjecture and they notice that there seem to be a lot of larger counterexamples. I think, it would be interesting to define a property P such that a. e. graph with this property is a counterexample. Even more interesting would be to prove that it can not be done for say 1st order definable properties P.*

CPH: *Among conjectures that passed their tests are: 766, 767, 768 and 773. In testing conjectures for cubic graphs they used programs of Gunnar Brinkmann. 5. 98.*

[L] Petr Lisonek, On a Conjecture of Graffiti, 92. Technical Report 92-67, Research Institute for Symbolic Computation, Johannes Kepler University, Linz, Austria)

[V] On Conjectures of Graffiti, V.

comp 753 and 840 and 862.

751. *Let G be regular graph with n vertices and c cut-vertices. If a is the independence number of G, then a is not more than n - c.*

Zoltan Furedi, University of Illinois at Urbana-Champaign, *proved this conjecture by showing that every d-regular graph with n vertices has at most n/2 cut-vertices. It is interesting, as he noticed, that this is not true for graphs with min degree d=3 , but nevertheless his proofs give that a + c is not more than n, for this class of graphs. These results are asymptotically close to the best possible, because it is easy to construct 3-regular examples in which both the independence number and the number of cut-vertices is about n/2. For min degree d > 3, Furedi showed* that the number of cut-vertices is not more than 2*n/d. January 93. Conjecture 757 describes another relation between cut vertices and the independence number.

752. Let us consider a random walk over vertices of a connected graph G with n vertices and let p(v) be the probability that at a given moment the walker is at the vertex v. If p = max p(v), then residue of G is not less than p*n. Residue is defined in conj. 69 which asserts that the residue is always smaller or equal to the independence number.

I do even know whether the independence number is more than p*n.

Prasad Tetali, ATT informed me that from the known results on random walks it follows that p(v) = deg(v)/n*average degree(G, and accordingly the conjecture can be restated in purely graph-theoretical terms as :

(*) independence number $\geq$ max deg/av. deg.

Colin Wright, Liverpool University sent similar information.

After the letter from Tetali, I found the following proof of (*) by induction on e = the number of missing edges of G. Let v be a vertex of max degree. If deg(v) $\leq$ n-1 we can add an edge, and v is again a vertex of max degree and (deg(v)+1)/av deg(G+u) ¿ deg(v)/av deg. If deg(v) = n-1, let G* = G-v. By Turan bound the independence number of G* geq (n-1)/(1+ av deg G*). But 1 + av deg G* $\leq$ av deg G, which proves (*).

(*) is sometimes much better than Turan's bound (for example in stars) and it would be nice to have a generalization of both of them. It is interesting though that (*) requires connectivity, which is used in Tetali's reduction. comp conj. 762. The conjecture about residue should follow similarly, but I have check it carefully. April 94.

753. *Let h be min h(v), where h is the number of v-horizontal edges of a connected graph G. Then the chromatic number of G is at most h +2. This is correct and it is slightly stronger than a conjecture made originally by Graffiti. The inequality probably can be improved when chromatic number goes to infinity.*

(*) If v is vertex and C and odd cycle of G then one of edges of C is v-horizontal.

Proof: Going around the cycle, the distance from v changes by at most 1. Since C is odd, two consecutive vertices must have the same distance from v.

Cor. The minimum number of edges which have to be deleted to make make a graph bipartite, is at least the maximum number of odd disjoint cycles. 95.

comp 750 and 840.

754. *The chip-firing game, [BLS] is a solitaire game played on vertices of a connected graph G. It starts with l(v) chips placed at each of vertices v of G. A vertex v whose degree(v) is not more than l(v) is called* ***loaded***. ***firing*** *of a loaded vertex v is sending one the chips of v to each of the neighbors of v, and then accordingly updating function l. A move in the chip firing game is selection a loaded vertex and then firing it. The game terminates iff there are no loaded vertices. Vertices which are not fired during the game are called* ***silent*** *G. Tardos proved that the game terminates iff at least one of the vertices of G is silent. Bjorner, Lovasz and Shor proved that if the game terminates then the final position and the number of times each vertex is fired does not depend on the order of moves and in particular the number of silent vertices is an invariant. They also proved that if the number of chips on the beginning of the game is not more than the e-1, where e is the number of edges of G then the game terminates. Suppose the chip-firing game starts with e-1 chips placed at one of the centers of G. This version of the game is closer to the game originally proposed by Spencer.*

Conjecture. *Let s be the number of silent vertices in the chip-firing game. Then the chromatic number of G is not more than the 2s.*

By an argument similar to that of Tardos, it is easy to show that in a terminating game every silent vertex has a silent neighbor (unless n=1) and thus a counterexample must have chromatic number at least 5, [V].

[BLS] A. Bjorner, L. Lovasz and P.W. Shor, Chip-firing games on graphs, Europ. J. of Combinatorics, 12(91), 283-291. [S] J. Spencer, Balancing vectors in max norm, Combinatorica, 6(86), 55-66. [T] G. Tardos, Polynomial bound for a chip-firing game on graphs, SIAM J. Discr. Math. 1(88), 397-398. [V]

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755. *Let a be the independence number of G, m - the matching number, h the length of the longest path and n the number of vertices. Then*

a - m + h is not more than n.

This conjecture is correct, but it seems to me of interest whether the decision problems concerning the size of a (or h) in the class of graphs for which we have equality are NP-complete.

A message from Odile Favaron:

I think that the connected graphs $G = (V, E)$ for which a - m + h = n can be described as follows:

$V = A \cup B$ with —A—¿—B—, $G[A]$ is independent, and G contains a path of length 2—B— (necessarily consisting of edges joining A and B). Then h=2m can be determined in polynomial time.

Odile Favaron, August 2002.

1.26.93

756. *The length of the longest path is $\geq$ than the global minimum degree. The Szekeres-Wilf invariant of a graph or its global minimum degree is the maximum of min degrees of all induced subgraphs of the graph.*

It is easy to prove a stronger statement: If G has minimum degree d then for every vertex v, there is a path of length d starting at v. Proof: let v-w be the longest path starting at v. If its length were shorter then d then w would be adjacent to a vertex not on the path which means that the path could extended.

757. *Let C be the set of non cut vertices of G, and let k be the number of components of C. Then the independence number of G is greater or equal than k.*

This is true about the number of components of any subset C, but even though obvious, the conjecture goes to the heart of the matter, because the independence number is equal to max number of components of C where max is taken over all induced subgraphs of G.

It is interesting that even though the program had a chance, it did not make similar conjectures about other subsets of G. The other bounds are probably much weaker and conjecture 751 may partially explain why Graffiti favored the set of non cut-vertices. Combined, these two conjectures provide both an upper and lower bounds for the independence number in terms of non cut-vertices. These bounds of course may be far apart, but that must be the case with all polynomially computable bounds for the independence number.

In general it would of interest to find other polynomial-time computable subsets of G, whose number of components provides a good lower bound for the independence number. In particular if would be nice to find such a set performing better than the set of non cut-vertices. One can also try to find a large independent set by consecutive removal of vertices whose deletion decreases connectivity of G. As a *measure of connectivity one could use the second smallest eigenvalue of the Laplacian of the largest component of G. Laplacian is defined in the next conjecture. It would be nice though to have also more intuitive measures of connectivity. Maxine the algorithm attempting to find a large independent set by deleting vertices of highest degree may be interpreted as such, because intuitively the vertices of high degree "connect" stronger than other vertices. Perhaps by introducing right measure of connectivity one can improve lower bounds for independence produced by Maxine, one of which is residue, see conj. 69.*

* * *

758. *The slowest expanding sequence of a graph G is defined as follows: We start with a vertex of minimum degree and assuming that d(1),d(2), ..d(k) were already defined, the next vertex is selected so that the set s = { d(1)...d(k+1)} spans as small set as possible. We stop when s spans all vertices of G. The span of a set W is the set of all vertices adjacent to one of the vertices in W. Expanding coefficients are numbers c(k) = min cardinality(span(X))/k, where the minimum is taken over all k-element subsets of G. Graffiti makes conjectures about expanding coefficients on the basis of the slowest expanding sequence of G. Conjecture: the smallest expanding coefficient of a connected graph G is not more than 1 + the second smallest eigenvalue of Laplacian of G. The Laplacian of G is the matrix enumerated by vertices of G, entries being -1 for adjacent and zero for nonadjacent pairs of distinct vertices. The entries on the diagonal are degrees of the corresponding vertices. This conjecture is false for large cycles, counterexamples can be found in [A] and [AM].* **Noga Alon, Tel-Aviv University**, *1. 93.*

[AM] N. Alon and V. D. Milman, lambda1 isoperimetric inequalities for graphs and superconcentrators, J. Combinatorial Theory, Ser. B 38(1985), 73-88. [A] N.

Alon, Eigenvalues and expanders, Combinatorica 6(1986), 83-96.

759. *Let us color vertices of a connected graph red and blue, to minimize the number of monochromatic edges. Suppose that the number of red edges is not less than the number of blue. Then the smallest expanding coefficient is not more than 1 + the number of red edges.*

760. *Let R be the set of red vertices from the previous conjecture and let $d(v)$ be the number of neighbors of v in R. Then the smallest expanding coefficient of a connected graph is not more than min $d(v)$ where the minimum is taken over all blue vertices.*

761. *Let E be the largest eigenvector of a connected graph G. E can be thought as function defined on vertices of G, and since it is nonnegative, it is a sort of a measure. The sum of values of E over vertices from a subset S, will be called the* ***spectral measure*** *of S. E is normed so that the sum of its components is the number of vertices of the graph, so for the regular graphs the spectral measure of S is simply the number of vertices of S, because in this case the largest eigenvalue is the degree of vertex. I do not know the answer to the conjecture below even for regular graphs.*

Conjecture. *The smallest expanding coefficient of a connected graph G is not more than 1 + spectral measure of a largest clique of G.*

* * *

May 93.

762. *Let $r(v)$ - the reciprocal degree of v, be the sum of reciprocals of degrees of neighbors of v. Then the independence number of G is not less than the maximum of $r(v)$.*

Proof: Consider the graph H induced by neighbors of v. Then the Caro -Wei bound (which is also called the Turan bound) implies that the independence number of H is already more than r(v). Caro-Wei bound is the sum of reciprocals of (1 + degree(v)). Let G be a graph such that the independence number in the graph induced by neighbors of v is a(v). Can we find a lower bound for the independence of G in terms of a(v), which would generalize Turan bound? comp 752.

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763. *The mean reciprocal degree = 1.*

* * *

June 5.93.

764. *An eigenvector can be thought of as a function defined on vertices of a graph. Let v be a vertex of G, let S be a complete set of eigenvectors of G, and let m(v) be the number of eigenvectors of G, whose value at v is zero.*

Conjecture: the independence number of of G is greater or equal to maximum of p(v). One problem with this conjecture is that it involves not just invariants, but what I call a try-out. In this case the correctness of the conjecture may depend on the choice of set S. I think that the best interpretation of a conjecture involving try-outs, is the strongest, which here would mean that conjecture is valid for all S. If the strongest interpretation is false, then sometimes, as it is the case here, it may be of interest if there is a set S satisfying the conjecture, particularly if one could define it.

June 14. 93.

765. *the length of the longest path is greater or equal to the average degree.*

Noga Alon *told me that this conjecture is correct and it follows from known cases of a more general conjecture of Erdos and Sos that a graph with n vertices* *and more than n(k-1)/2 edges contains every tree on k edges, [B]. This is known for various trees, like stars, paths and fishbones (caterpillars). For paths this gives 765. Nov. 93.*

[B] Bela Bollobas, Extremal Graph Theory,

766. *Let d(v) ve the number of vertices at odd distance from v and let h(v) be the number of v-horizontal edges whose both endpoints at odd distance from v. If G is a cubic graph then the independence number of G is greater or equal to min d(v) - min h(v).*

The program made also the "even" version of this conjecture, i.e with "odd" being replaced by "even" in definitions of d and h. comp 750. This and a few next conjectures were tested on about 600 hundred small cubic graphs which I received from Gordon Royle.

The smallest counterexample to this conjecture must have at least 21 vertices, see **CHP, conj 750***. 5. 98.*

767. *Let e(v) be the number of vertices at even distance from v and let let h(v) be the number of v-horizontal edges whose both endpoints are at even distance from v. Conjecture: If G is cubic then the independence number of G is greater or equal to the average value of e(v) minus the average value of h(v).*

The smallest counterexample to this conjecture must have at least 21 vertices, see **CHP, conj 750***. 5. 98.*

768. *Let e(v) be the number of vertices at even distance from v and let let h(v) be the number of v-horizontal edges.*

Conjecture: If G is cubic then the independence number of G is greater or equal to the average value of e(v) minus min h(v).

The smallest counterexample to this conjecture must have at least 21 vertices, see **CHP, conj 750**. *5. 98.*

769. *Let m = max e(v), where e(v) is the number of vertices at even distance from v. If G is a cubic graph then then the independence number of G is not more than m.*

Gilles Caporossi, **Pierre Hansen**, *and* **Florian Pujol** *found an 18-vertex counterexample, see 750.*

770. *Let m be the same as in conj. 769. If G is cubic then the independence number of G is greater or equal to (1 + m)/2.*

771. *The independence number of a cubic graph G is greater or equal to the number of nonnegative eigenvalues of G - diameter of G.*

The conjecture is false. Counterexamples are large random cubic graphs. Their maximum independent sets have size less than (0.5-c) n, where c$\geq$0 is independent of n, they have about n/2 positive eigenvalues and diameter is O(log n),[BDM]. **Noga Alon,** *Nov. 93.*

[BDM] Brendan D. McKay, The expected eigenvalue distribution of a large regular graph, Linear Algebra and its Applications 40 (1981) 203-216.

[Note : the formula for F on p213 should have 1/2 not 1/4, Brendan McKay.]

772. *An interval [u,v] in a graph G, is the set of all vertices x, such that d(u,x) + d(x,v) = d(u,v), where d is the distance function in G. If G is a cubic graph then the independence number of G is not more than 1 + the number of vertices in a maximum interval of G.*

Gilles Caporossi, **Pierre Hansen**, *and* **Florian Pujol** *found an 20-vertex counterexample, see* **CPH** *in 750.*

773. *The radius of a cubic connected graph is not more than 0.5(n-residue).*

Residue is defined in conj. 69. For regular graphs of degree d, residue is [n/(d+1)], where [x] is the smallest integer greater or equal to x, [FMS]. The conjecture is probably not difficult, but it may be of some interest because of conjecture 0, (that radius is not more than independence number) and the notorious occurance of the independence ratio 3/8 in many questions for cubic graphs.

The smallest counterexample to this conjecture must have at least 21 vertices, see **CHP, conj 750**. *5. 98.*

774. *The independence number of a cubic graph is greater or equal to the number of its eigenvalues strictly greater than 1.*

For large random cubic graph, the number of such eigenvalues is about n(1-F(1)), where F(1) (see [BDM], p. 213, conj.771.) is the number of eigenvalues less or equal to 1. Approximately 1-F(1) = 0.34740645. This is more then the independence number of these graphs and thus the conjecture is true for almost all cubic graphs. **Brendan McKay,** *11.93.*

Gilles Caporossi, **Pierre Hansen**, *and* **Florian Pujol** *found an 18-vertex counterexample,* **CPH** *in 750.*

775. *Let f(v) be the number of times of firing vertex v in the chip firing game (see conj. 754) If G is cubic then diameter of G is not more than max f(v).*

776. *Let p be the sum of positive eigenvalues of G. If G is cubic then the independence number of G is greater or equal to -1 + p/2.*

Gilles Caporossi, **Pierre Hansen**, *and* **Florian Pujol** *found an 18-vertex counterexample,* **CPH** *in 750.*

777. *If X is a set of vertices of a graph then Sp(X) is the set of all neighbors of elements from X. An independent set X is called a counter-independent set, if the complement of Sp(X) is independent. A jet is a counter-independent set X such* *that the complement of sp(X) is a maximum independent set. The cardinality of a smallest counter-independent set is called the counter-independence number of G, and the cardinality of the smallest jet is called the jet number of G.*

Obviously if we know a minimum jet set then we can find in a polynomial time a maximum independent set. In particular the problem of finding minimum jet sets is obviously NP-hard, but I do not have at the moment any proof that the decision problem: "given G and k, is the jet number of G smaller or equal to k," is NP-complete. The jet number is of course never greater than the independence number and often is much smaller as for example it is indicated in the following

Conjecture: If G is a cubic graph then its jet number is at most [n/4], where [x] denotes the smallest integer greater or equal to x.

In addition to this conjecture of Graffiti I have one of my own: for almost every graph G, the jet number of G is not more than half of the independence number of G. To be more precise I conjecture this for random graphs. It is interesting that the conjecture is not true for Paley graphs. In Paley graph with 61 vertices, both the independence and the jet number are equal to 5 (according to my programs.)

First **Noga Alon** *and then* **Bela Bollobas** *told me (about two years ago) that my conjecture is likely to be true. It would be then interesting to know if the jet number of large Paley graphs can be s ignificantly bigger that logn, 3. 96.*

Jet numbers may be much smaller than the independence numbers as it is the case with stars. In general, it is easy to show that the jet number is never more than n/2.

See few conjectures below for other bounds.

Let IP be the following integer programming problem:

For each edge (i,j) we have two variables x_{ij} *and* x_{ji}.

For edge (i,j) we have a constraint

(i) $x_{ij} + x_{ji} \leq 1$.

Let $R(i) = sumx_{ij}$, *and* $C(i) = sumx_{ji}$

where the summation is over all j *adjacent to* i.

The next groups of constraints are

(ii) $R(i) \leq 1$ *and* $C(i) \leq 1$ *for each vertex i,*

(iii) $R(i) + R(j) \leq 1$ *for each edge (i,j), and*

(iv) $R(i) + C(j) \leq 1$ *for each edge (i,j).*

Remaining constraints may assure that all variables are $0's$ *and* $1's$, *and that non-zero variables induce a matching. The objective function to be maximized is the sum of all variables.*

A solution of IP defines a maximum polar matching. 12. 96.

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778. *The jet number of any graph is not more than the number of positive eigenvalues of G.*

This and next two conjectures were proved by **Ermelinda DeLaVina** *and myself, [DLV], and the same proof shows that the jet number is not more than the number of negative eigenvalues of G.*

[DLV] Ermelinda DeLaVina, An Investigation of Counter-Independence and Jet Numbers of a Graph, Master Thesis, University of Houston April 93.

779. *For every connected graph G, the counter-independence number of G is greater or equal to half of the radius, [DLV].*

780. *The jet number of any connected graph is greater or equal to half of the radius.*

Before, as listed 780 was stating that counter-independence is less or equal to half of the radius, but Graffiti could not make this conjecture, and I remember that I was working on 779 for jets.

781. *Let G be the global minimum degree of the complement of G. Graffiti conjectured that the jet number of G is not more than 1 + g/2.*

DeLaVina, *[DLV] found a very nice counterexample which I'll describe in more general terms particularly that they may be related to my conjecture from 777.*

Let F be a family of subsets of set X. The ***starfish*** *S(F,X) is the graph whose vertices are elements of X or F, and two distinct vertices u,v are adjacent iff either u,v are elements of X, or v is an element of u which is an element of F or u is an element of v which is an element of F.*

DeLaVina's example is S(F,X), where X is a set with 5 or more elements and F is the family of all (m-2)-element subsets of X. Starfishes proved to be very useful counter-examples for other early conjectures of Graffiti about the jet numbers. They naturally arise in these problems because of Theorem 3 of [DLV].

Let S(m) be the starfish S(F,X), where X has m elements and F is the family of all m-1 element subsets of X. What is the largest integer m = n(m) so that the random graph R(n) on n vertices, (p=0.5) almost certainly contains an S(m) as a (not necessarily induced) subgraph. To prove (mine) conjecture from 777 it is enough to show that m is not more than logn.

DeLaVina proved that the unless G is a complete graph than its jet number is at most not mare than Szekeres-Wilf Invariant (the global min.), 9. 96, [DLV1],

[DLV1] Ermelinda DeLaVina, On Graffiti's Conjecture 781 about Jets of Independent Sets and Szekeres-Wilf Invariant.

10.93.

782. *Let $s(k)$ denote the minimum cardinality of sets spanned by all k-element independent sets X of vertices. Sp(X) is the set of all vertices adjacent to one of the elements of X. Then the jet number is not more than (n- s(k) + k)/2. Graffiti made this conjecture for k = 2, but it is true for arbitrary k smaller then the jet number.*

2. 98.

782a *If the independence number of a connected graph G is 2 then G is hamiltonion.*

I think this conjecture was never listed here, but it is discussed in [V], (Corollary 3.8). I just noticed in Lovasz's "Combinatorial Problems and Exercises" the following result: If α is the independence number of G then vertices of G can be represented as a union of α vertex-disjoint paths.

Here is a generalization of both statements:

If G is a connected graph which is not complete then G is a union of $\alpha - 1$ vertex disjoint paths.

Proof: Let P be a maximal path of lenght ≥ 2 which does not induce a cycle. P exists, since G is not complete. Let H be the graph induced by G P. Then the independence of H is at most $\alpha - 2$, and the statement follows by induction.

[V] Siemion Fajtlowicz, On Conjectures of Graffiti, Part V. Graph Theory, Combinatorics and Applications, 7th Quadrennial Conference on Theory and Applications of Graphs. vol 1. 367-376. 2. 98.

* * *

783. *If S is a set of integers then AP(S) is the graph whose vertices are elements of S, two being adjacent iff they are contained in a 3-term arithmetic progression.*

Let [k,n] be the set of primes less than n and more than k. The first conjecture of Graffiti about these graphs was essentially that

(*) the minimum degree of AP([2,n]) is equal to its minimum cut.

The actual conjecture was verbally the same but, its meaning was somewhat different, because the version of the program which conjectured (*) was designed for connected graphs. The data seem however indicate that if $k \geq 9$ then the graphs AP[k,n] are connected. Looking up values of minimum degree for graphs AP[2,n] I could not miss the following conjecture:

(**) Every odd prime is contained in a 3-term arithmetic progression of primes (ap) Actually the data seemed to indicate that there must be a lot of such triples for every prime. The next day I stated (**) in my class and added that these progressions are probably as long as one wishes. If p is large enough than perhaps, but **Ermelinda DeLaVina** *pointed at once that a progression starting with p may have at most p terms. It is easy to find 5 and 7-term progressions of primes starting respectively with 5 and 7 (the differences are 6 and 150), but at the moment I do not know any 11-term progression starting with 11. The longest I found, had length 8 and the difference 1210230. I was certain that (**) is a well known fact or a conjecture, but Professor Erdos, told me that he never heard of it. Nov. 93. comp 784 and 785.*

*The difference 2*9*5*7*51859 produces a 9-term app starting with 11. I found it searching only through differences divisible by 2*3*5*7. s.f Dec 93.*

*(***) for every prime p there are infinitely many p-term app starting with p, with the difference of the form q*Q, where Q=1 or Q is a prime, and all prime factors of q are less than p. Moreover*

(****) each prime less than p is a factor of q. The last condition is necessary. Indeed if r is a prime less than p, then (unless q mod r = 0), q, 2q, .. (r-1)q have distinct, non-zero residues mod r and thus there is $k < r$ such that p + kq = 0 mod r.

[ME] contains an intriguing and stronger than 3000 dollar conjecture of Erdos. The longest app of primes (which was known in 91) has 19 terms,[PR].

Micha Hofri *found an 11-term app starting with 11 and the*

*difference 16*3*5*7*13*37*1901. Dec 93.*

I also conjecture that

(v) for every k there are k-term arithmetic progressions of consecutive primes. They are easy to find for k = 4, but the largest k for which I know that the conjecture is true is 5. The progression starts with 9843019 and has the difference 30.

Andrew Odlyzko, ATT told me that the (v) is a special case of a conjecture of Hardy and Littlewood.

2.11.94. 9. 97. It is interesting that I had (at the time) a conversation with a mathematician who said that this conjecture is so outrageuos that it must be false. 9. 97.

I mailed 783 to **W. Holsztynski, Wiltel Corporation***, yesterday at midnight, and at 4 am today, I received number of propositions related to the case p=11. There are too many of them to be included here. Some of these results are similar to conditions (D) below.*

Partially because of Holsztynski's letter and partially because of the idea (of the sequence) defined in 784, I wrote program ***Primate*** *which examines one by one all even integers d as candidates for the difference in app starting with p. Both Primate and procedure described in 784 act as sort of dynamic sieves. As in 784*

*Primate could be described as pair of "interacting" sequences but this time I prefer to describe the process as a program: If d produces an app of length p then the corresponding sequence is printed by Primate (d is accepted.) Otherwise there is a prime q and integer k such that (p + k*d) = 0 mod q. Let d(q) = d mod q.*

All future candidate d's accepted by Primate must satisfy the condition

(D) d mod q $\neq$d(q).

Primate adds the pair (q,d(q)) to the list of forbidden conditions (starting with empty list) to speed up its performance and it conspicuously does so. Looking at the data generated by the first run of Primate for p = 3, I again could not miss the following conjecture: The number of accepted d's grows much faster than the number of forbidden conditions and the same seems to be true for p=5. Actually it seems that the ratio goes to 0. Conjecture: Let d(n) be the number of d's accepted by Primate at the time n, and and let c(n) be the number conditions. Then for every prime p, we have that c(n) = o(d(n)).

2.17.94

Holsztynski made good of one of his results by finding in about 2 min, the 11 term app found before by Hofri. Hofri's program took several days of CPU, but was run on the Next using Mathematica. Holsztynski used more straightforward primality tests but run his program on much faster Alpha. Within the next two hours of CPU, Holsztynski found two new 11-term progressions. The differences are 4911773580 and 25104552900.

2.19.94

Hofri *implemented Primate and found the progression for p = 11 in less than 7 min of CPU on Dec 3100.*

Both **Holsztynski and Hofri** *found a 12-term app starting with 13. The difference is 1482708889200. They also found a 12-term app starting with 17. For this one the difference is 154926241470. Then Hofri found the 2nd 12-term app starting with 13. The difference is 4375240131570 = = 2310*2729*99149. Hofri and Holsztynski use the same machines and basically the same ideas as above, but they gradually make various improvements.*

2.24.94.

Holsztynski *found the first 13-term app starting with 13. The difference is 9918821194590. He also found 30 11-term app's starting with 11. Holsztynski found two more 13-term app starting with 13. The differences are 187635245859600 and 232320390245790, the second one is about 20 times larger than the first one. March 94.*

[PR] P. A. Pritchard, Long app's; some old, some new, Mathematics of Computation, 45 (1985) 263-267

[RI] P. Ribenboim, The book of prime number records, Springer NY 88.

[ME] N. MacKinnon and J. Eastmond, An attack on the Erdos conjecture, The Mathematical Gazette, 71 (1987) 14 - 19.

784. Partially, because of (*) from 783, and partially because the sequence defined below seems of interest for its own sake, I decided to try to obtain conjectures related to the twin prime conjecture. Graffiti generated quickly a whole bunch of them, but so far, all of them seem to be at least as strong as the twin prime conjecture. To generate conjectures, I defined the following two sequences of numbers: Let q(1) = 2, and let p(1) = 7. Assuming that p(n) and q(n), were already defined, q(n+1) is defined to be the smallest prime dividing 2+ p(n). p(n+1) is then defined as the smallest prime $p \geq p_n$, *such that p+2 is not a prime, but for every* $k \leq n+1$, *p+2 is not divisible by q(k). q(3) is 3 and p(3) is 23. Professor Erdos and Craig Larson conjecture that every prime occurs as a q(n). The sequence of p(n)'s grows very fast, and as a matter fact Professor Erdos proved that the series of reciprocals of* p_n *converges. The idea of his proof is based on the fact that 2+p(n+1) must have at least two factors, each at least as large as q(n). Let* $d_n = p_{n+1}$ - p_n, *and let* M_n = *max*$\{ d_k : k \leq n \}$. *An example of Graffiti's conjecture about these sequences is that the number of twin primes smaller than p(n), is at least* $(1 + M(n))^{0.5}$. *[HW] contains a conjecture that the number of twin primes not more than n is O(n/((logn)*^{2}*) and they even conjecture the exact value of the constant. Nov 93.*

[HW] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford Science Publications,

December 93

785. Let d be the maximum degree of the graph AP[10,n], defined in 783 and let l be the maximum eigenvalue of the Laplacian of this graph. Then d is not more than -1 +l.

The conjectured bound seems to be very tight for n $\leq$ *250. Professor Erdos told me that the density of 3-term a.p.p starting with p, should be asymptotically the same as the density of twin primes. It may be therefore of interest to know how this conjecture is related to the conjecture of Hardy and Wright described in 784.*

June 14 94.

In conjectures below, r(3,a) will denote the class of all largest triangle-free graphs with the independence number equal to a, a $\geq$ *2. Thus r(3,a) are critical Ramsey graphs R(3,a+1). I am grateful to Staszek Radziszowski for sending me a large collection of r(3,a) graphs a part of which I incorporated into Graffiti. The collection includes all critical 161 R(3,7) graphs and 4 of R(3,8). At the moment I* *used about fifty of these graphs, but later I may test conjectures against remaining graphs. There is only one known R(3,9) graph, and the known bounds for R(3,10] are 40 and 43 respectively, [R]. Nevertheless Geofrey Exoo told me that R(3,10) is probably 40, so perhaps it make sense to take a look at 39-vertex examples as possible counterexamples to conjectures. If A is set of vertices of a graph, and v a vertex then the* ***coordinate*** *of v (with respect to A) is the number of neighbors of v in A. Coordinates are generalizations of degrees. Given A,* $a_k = a_k(A)$ *is the number of vertices whose A-coordinate is k. In all conjectures below involving coordinates, A is a maximum independent set. Usually there are many such sets and a conjecture can true be for some them and false for others, i.e, conjectures can be subject to several interpretations. So coordinates are not invariants, but what I call tryouts. The current version of Graffiti selects for making conjectures a maximum independent set realizing the jet number of G (see conj. 777). Accordingly, unless it is explicitly stated, conjectures involving coordinates are primarily about these sets, But even they are far from being unique. It may be of interest whether they are true for all maximum independent sets, those which realize the jet number, or at the other extreme it may be of interest, whether there is just one independent set satisfying a conjecture. In general I am in favor of the strongest interpretation of a conjecture, but in the case of Ramsey graphs the main question is existence.*

Conjecture 786 was first made by Graffiti on the basis of a very few critical Ramsey graphs and later when Radziszewski sent me his graphs some of them, but rather very few, turned to be counterexamples to the strong interpretation of the conjecture. The weaker forms still may be true, but I include the conjecture mainly because the involved invariants show what seems to me an unexpected pattern. If *there is anything to it, then perhaps it may help, both in proving the upper bounds for R(3,a), as well as finding new examples by computer search.*

[R] S. Radziszowski, Small Ramsey Numbers, preprint.

786. *If G is a critical Ramsey graph r(3,a) then the jet number of G is equal to* a_1.

Out of about fifty Radziszowski's graphs, used in the first run, all but two satisfy the conjecture, so it is still possible that always there is a maximum independent set with respect to which the conjecture is true.

The pattern mentioned above is that for all of these graphs a_1 *is very large.*

For about half of r(3,6) graphs a_1 *is 5, and for about half it is 4. For two of them it is 3. The jet number for these graphs is always 5 or 4. In all 4 of Radziszowski's r(3,7) graphs both jet and* a_1 *are 6. Note that since our graphs are triangle free, and coordinates are with respect to a maximum independent set, therefore* a_1 *is never more than a. Similarly the jet number is always greater or equal to* a_1*. The only two (known to me) examples where* $a_1 = a =$ *jet number, are R(3,4) critical graphs with 10 edges, and the R(3,5) critical graph. Perhaps they are unique. To verify it, should be much easier than finding critical Ramsey graphs, because the property seems to me very strong. This may also be a good testing ground to see what happens in the case when a(1) is very large, which so far seems to be a sure bet. Even if there are no other critical Ramsey graphs with* $a_1 = a$*, the graphs with this property seem of interest as the extreme cases of inequality that the jet number is not more than a.*

Let max_{a_k} *denotes maximum of* $a_k(I)$ *over all maximum independent sets I. It turns out that in all but the regular 8-vertex r(3,n),* max_{a_1} *is equal to the independence number. One does not have to be a disciple of Dr. Indiana Jones to make* *an appropriate conjecture, and perhaps I'll adapt Graffiti to make conjectures about a_k with respect to these special independent sets I. Because of the conjecture 788, I plan to impose an additional condition, namely that I maximizes a_2 (subject to the condition that $a_1 = a_0$, i.e the independence number. 6.21.94*

787. *If G is a complement of a r(3,n) graph then the number of negative eigenvalues of G is equal to the minimum degree G.*

788. $a_2 \geq a_3$.

On the basis of this and other similar conjectures, it looks that a_2 will often dominate other a_k. If this is correct and particularly, if it can be assumed on top of the condition that $a_1 = a_0$ for a MIS, then we can form guesses about a fairly large subgraph of r(3,n). The guesses can be based on the following remarks:

1. Let v be a a_2 vertex and r_1 and r_2 its neighbors in I. Let b_1 and b_2 be the unique a_1 vertices adjacent to r_1 and r_2 respectively. r's and b's will be called respectively roots and branches of v. b_1 and b_2 are adjacent in G, and in particular v, r_1,r_2,b_1, b_2 form an induced pentagon.

2. Different a_2 vertices can't have the same set of roots. If they have a root in common then they are not adjacent.

Let A_k denotes the subgraph induced by a_k vertices. On the basis of 1 and 2, I would conjecture that the structure of A_2 is to a large degree determined by the structure of the complement of the line graph of A(1).

Hopefully Graffiti can conjecture more when I'll run it for MIS with $a_1 = a_0$ maximizing a_2.

I will ask now a question which can be tested with computers.

Let R be a triangle-free graph on an n element set V. For each v from V, let us add a vertex v and join it to v. Let us call the new graph R*. Next let us form a* *disjoint union of R^* and $L(R)$ and join every vertex (a,b) of R^* to a^* and b^*. Let us call this graph R^{**}.*

For what graphs R, there is a critical Ramsey graph R(3,n) containing R^{**} as a subgraph, so that R^* forms a maximum independent set, R is the set of its a_1 vertices and L(R) is contained (or even better equal) to the set of its a_2 vertices.

Note that condition 2 impose some adjacency relations for a_2 vertices, and additional conditions follow from

3. if a and b are a_2 vertices with a common root, then a is adjacent to a branch of b, or b is adjacent to a branch of a. In particular a and b together with their roots and branches induce an R(3,4) critical graph.

789. *The market value of critical Ramsey triangle-free graphs is equal to the number of eigenvalues strictly greater than -1. (749.)*

The market value of these graphs seem to be n - maximum degree.

comp 790 below.

790. *The independence number of critical Ramsey triangle-free graphs is equal to their maximum degree.*

I think that this conjecture is false, but it would be interesting to know a counter-example. On the other hand perhaps one can prove that for every n there is a critical Ramsey graph R(3,k) with the independence number equal to k.

comp. 789.

I found that the following related problem of Erdos is open: Can one find a

(*) regular graph on n vertices with no triangles whose largest independent set is equal to its maximum degree, [A], p. 222.

For what n there are graphs satisfying (*) whose chromatic number is also equal to the maximum degree? Note that maximal R(3,4) and R(3,5) graphs have *this property. I think that there should be only finitely many such graphs. Are there arbitrarily large graphs of chromatic number k satisfying Erdos's condition (E). 2. 96.*

Yair Caro, Haifa University, Israel *noted that the product of a graph G satisfying (*) and an empty graph E_k also satisfies (*). The product, in this case, is obtained by replacing every vertex of G with a k-element independent set and joining two vertices by an edge if their projections onto G are adjacent. 2. 96.*

I checked that there are no 9-vertex graphs satisfying (*), but the cyclic Cayley graph on 11 vertices with the generating set 1, 3, 8, and 10, has the property. I think that one should have these graphs for every prime $p \geq 7$, and thus by Caro construction for every n, apart from some small powers of 3.

(**) What happens if instead of triangle-free we ask Erdos' question for graphs of girth k, $k \geq 5$. see (***)

Let d be the degree of a graph satisfying (*). d is at most n/2, and at least $(n-1)^{1/2}$, because the diameter must be 2, i.e., for n = 11, the only possibility is d = 4. 2.96.

Caro noticed that apart from d = 2, obviously there are no graphs satisfying (*). 4. 96. I clearly should have ask the following version of Erdos question:

(***) for what d and n are there d-regular n-vertex graphs of girth 5 with the independence number 2(d-1). For girth 6, the obvious bound is 1 + d(d-1), etc. For each of these two cases the respective examples are the Petersen and the Heawood graph, and I think that it should be possible to characterize all graphs with equality.

Df. A sequence $l_1 ... L_d$ of sets of vertices is an **alining of G** *and its elements are called* **lines** *if*

1. lines are independent subsets of G, and

2. If J is a k-element independent set of vertices of G then J meets at least k lines.

An alining is **d-regular** *if*

3. every vertex v of G belongs to d - deg(v) lines.

Before the simple theorem below, I considered alinings in connection with inner graphs in 863, but at the time, the concept did not seem enough motivated. Instead of assuming that elements of alinings are independent, we can assume that they induce subgraphs belonging to a class of $K-s$*-free graphs or even in general arbitrary class of graphs, but that goes beyond this conjecture.*

Theorem: Let G be a graph d-regular satisfying (*), v - a vertex of G, N - the set of its neighbors, and M - the graph induced by vertices of distance 2 from v. Let P be the family of neighbors of vertices of N in M. Then P is a regular alining of G consisting of maximum independent sets of M.

Conversely if (M, L) is a regular d-alining of triangle-free graph M consisting of maximum independent sets, then there is a unique graph G satisfying (*), such that M is isomorphic to the graph induced by G - N - v.

Proof: The second condition is a consequence of (*). If an independent set J of M were contained in a too few lines induced by a subset K of N, then N - K + J, would be an independent set with more than d elements. An independent set in M has at most number d-1 vertices, because together with v it forms an independent set in G. The other conditions are obvious.

To get the converse we define G = E(M) to be M + L + singelton with the obvious adjacency relation. Since M is triangle-free then so is G. By 2, the independence number of G is d, and by 3 implies that G is d-regular.

Examples:

1. Let G be an 8-vertex graph satisfying (*). If d is 4 then G is $K_{4,4}$ *and the corresponding is a constant sequence. If d = 3, then N has 4 vertices and 3 edges. The independence number of M must be 2, i.e., N must be isomorphic to* P_4*. Since d=3 and there are exactly 3 maximum independent set, M has a unique alining, which determines the unique regular Ramsey graph R(3,4), (one just have to notice that no line can occur twice.)*

Pentagon - the critical R(3,3) graph arises similarly from K_2*.*

2. Let G be the Ramsey graph R(3,5). Selecting a vertex we get a 4-pasting of M - the minimal Ramsey graph R(3,4), because the graph induced by M must have 10 edges. The elements of this pasting are maximum independent subsets of M containing exactly one vertex of degree 3.It is fairly clear that this a unique pasting of M leading to R(3,5).

[A] Applications of Discrete Mathematics, Siam 88.

791. *A span of a vertex of critical Ramsey graph is never contained in a span of another vertex.*

April 95.

792. *The largest independent set of a graph G is not more than 1 + the largest eigenvalue of the complement of G.*

This conjecture is true because of Cauchy's Interlacing theorem applied to a maximal clique of the complement of G. Since the maximum degree is n - 1 - minimum degree of the complement, and the maximal eigenvalue is not more than the maximum degree, this conjecture is stronger than 158.

comp 793.

793. *The largest independent set is not more than 1 + global minimum degree of the complement of G. (756.)*

This conjecture is stronger than 792 and it is correct because the 1 + global min is an upper bound for the chromatic number and hence for the clique number of the complement. comp. 781.

There are many graphs for which equality holds true. I do not have a characterization of these graphs.

794. *Let $v = v_0..v_n$ be a nondecreasing sequence of integers. The derived sequence v^* is defined by deleting from v the first 1 + v_0 terms (or all of them if n is less than 1 + v(0)). The number of iterations until the sequence is empty is called the lower quotient of v.*

The upper quotient is defined dually, i.e we start with v sorted in nonincreasing order, delete the first 1 + v_0 terms, etc.

Conjecture: The lower quotient of the degree sequence of a graph is not more than its independence number.

The conjecture is correct but I do not have characterization of quite a few graphs for which we have equality.

comp. 795 and 796, which are easy to prove.

795. *The lower quotient of the degree sequence is not less than the Turan bound. (69,794.)*

796. *The upper quotient of the degree sequence is not more than the Turan bound. (69, 794.)*

797. *Turan bound = 1 + the average temperature of the complement of G.(-1,69.)*

As a consequence Ramsey Theorem can be stated as follows: for every real t, there is n such that every graph of order n contains an induced subgraph of temperature t, or its complement contains an induced subgraph of temperature t. Too *bad that the maximum temperature will be always an integer, because we could have fractional Ramsey numbers. Perhaps in some generalizations.*

798. *Let* $p = 4k + 1$ *be a prime. Let* a_k, *k = 1,..p-1, be r or n depending on whether k is or nor a quadratic residue mod p. Then the number of consecutive rr pairs of the form* $a_{2k+1}, a_{2k})$ *is the same as the number of consecutive nn pairs. Similarly the number of consecutive rn pairs is the same as the number of consecutive nr pairs.*

This conjecture illustrates the point made in the comment to conjecture number -1. **Andrew Odlyzko** *solved it at once using well known (to him) tools from the analytic character theory. I enclose his e-mail here with underst his kind permission:*

" It's interesting that Graffiti found this result. However, it is an easy consequence of known results. There may be a more direct way to see it, but here is the proof I found right away:

The number of rr pairs is 1/4 times

*sum((1 + chi(2*k-1)) * (1 + chi(2*k), sum on k from 1 to (p-1)/2),*

and for nn pairs is similar but with two -'s in the sum in place of the two +'s. Hence to prove the two are equal, it suffices to show that

*sum(chi(2*k-1) + chi(2*k), sum on k from 1 to (p-1)/2)*

is equal to its negation (i.e., that it is 0). However, the sum above is just

sum(chi(k), sum on k from 1 to p-1),

which is clearly 0.

P.S. chi(k) is the quadratic character of k."

After I understood the proof, I realized that the stronger, obvious statement is true:

(*) Let us color elements of a 4k-element set S red and blue so that the number of blue elements is the same as the number of red. Then for any linear ordering of elements of S the number of consecutive red pairs (in which the first element is odd) is the same as the number of such blue pairs.

I do not know how to generalize the principle (*) to a larger number of colors.

798 is one of the first conjectures of a new version of Graffiti which makes conjectures about arbitrary sequences of symbols, and in particular about the Griggs-Waterman double digest graph of DNA sequences. Temporarily I had to abandon this version, but hopefully in a near future I'll include here some of its conjectures. One problem with this new version is that there is no satisfactory mathematical model of DNA sequences, so it is not clear how to deal with these conjectures. I discuss this difficulty in [ACC].

Nevertheless Odlyzko told me about some recent work which perhaps will make it plausible,[6]. The paper though strives toward a statistical rather than strictly mathematical model. I would appreciate any information related to this problem; for the purpose of making of conjectures it would be quite enough to have a definition of special class of DNA sequences.

[6] Martin Farach, Martin Noordewier, Serap Savan, Larry Shepp, Adi Wyner and Jacob Ziv, Finding Splice Junction, Information Theory, Asymptotics and Nature, preprint 94.

799. *Let S be a set of vertices of a graph G = (V,). The S-quasi-ordering of vertices of G is the relation Q(u,v) iff u=v or N(u) - the set of neighbors of u in in S is a subset of N(v). Let E(u,v) iff Q(u,v) and Q(v,u). The S-poset of G is the induced factor-ordering of the equivalence classes of E and it is denoted by PS(G). If S=V then (V/E,Q/E) is called the poset of G and it is denoted by P(G)*

Conjecture: The independence number of G is greater or equal to the number of elements of a maximum principal filter of the quasi-ordering of G.

This conjecture is a first solution to Invariant Interpolation Problems which will be described in 814; apart from conjectures Graffiti now ask very specific question of a certain type, answers to which may help it to make better conjectures.

May, 95.

Let S be a set integers, and G = PR[S] the graph whose vertices are elements of S, two being adjacent iff they are not relatively prime. Conjectures 800 : 813 are about graphs of the form PR[S], where S is set of square-free integers from the interval [2..n]. Graffiti made them on the basis of all $n \leq 100$ and another 20 or so $n \leq 200$, with exception of conjectures involving the jet and the counterindependence number. Those are made on the basis of $n \leq 42$.

800. *The independence number of G, (i.e the number of primes from S) is not more than 1 + the number of nonpositive eigenvalues of the complement of G.*

The conjecture is valid for all graphs, and as matter of fact I seem to remember that in the past Graffiti made a stronger conjecture in which "nonpositive" was replaced by -1. But recently the Dalmatian version, [V] might skip this conjecture, as less informative than conjectures which it already knew. On the other hand the data indicates that this conjecture is a very good bound for PR-graphs, perhaps because nonnegative eigenvalues were not included in this run and that's why it was made here. Of hand I think that "-1"-version of the conjecture must be closely related to the correct part of 446. the remark can be used as an example of a solution of an IIP

The conjecture follows in the standard way, from Cauchy's Interlacing Theorem, because cliques with n vertices have n-1 eigenvalues equal to -1.

801. *Every performance of Maxine and MIN, produces a maximum independent set in G.*

MIN is essentially equivalent to Eratosthenes sieve (in reversed order.)

Let I be a maximum independent set of vertices of an (arbitrary) graph H and let N(v) be the set of neighbors of v in I. If H has the property that for every u not = v, u is adjacent to v iff N(v) and N(u) intersect, then H belongs to the domain of Maxine and MIN.

Proof: If if x belongs to N(v) then degree of N(v) is more than degree of x, because v is adjacent to every vertex y adjacent to x.

* * *

802. *the independence number is not more than the Turan bound + the largest eigenvalue - the second largest eigenvalue.*

Turan bound is not more than the independence number and for these graphs it is likely to be close to the residue of PR[2..n]. A very good estimate of the latter was obtained by Erdos and Staton, see conj. 470.

803. *The Euler phi function f of a graph is defined as follows: Let I be a maximum independent set of vertices. Let N(v) be v + the set of neighbors of v in I. f(v) is the number of vertices u such that N(u) and N(v) are disjoint. The phi function is what I call a tryout, rather than invariant because it depends on the choice of I. For PR[S] however phi function is an invariant.*

Conjecture: the independence number of G is not more than the residue of the complement of G + the minimum phi function of G.

804. *the independence number of G is greater or equal to the upper quotient of the degree sequence + the number of eigenvalues greater or equal to 1.*

805. *the largest eigenvalue of G is not more than 1 + sum temperatures of vertices of G.*

806. *the largest eigenvalue of G is not more than the number of vertices of different degrees.*

807 *the second largest eigenvalue is not more than half of the largest eigenvalue.*

808. *The largest eigenvalue of G is greater or equal to the mean dual degree.*

809. *The number of positive eigenvalues is not more than -1 + residue.*

810. *The second largest eigenvalue is greater or equal to the product of global minimum degree and the counterindependence number.*

811. *The second largest eigenvalue is greater or equal to the product of the frequency of maximum degree and the jet number.*

812. *the largest eigenvalue - the second largest eigenvalue is not more then standard deviation of the degree sequence + k/l, where k is the number of negative eigenvalues and l the number of positive eigenvalues.*

Both sides of the inequality seem to be very close.

813. *the largest eigenvalue - the second largest eigenvalue is not more then standard deviation of the degree sequence + mean temperature.*

June 95.

814. *Apart from conjectures the performance of Graffiti suggests now certain questions, which I call Invariant Interpolation problems. The first and so far the only interpolation problem which I was able to solve is this:*

Consider the graph RP[n], whose vertices are numbers 2..n, two being adjacent iff they are relatively prime. The problem is to find an easily, say polynomially computable graph-theoretical invariant I, such for every graph G, the independence *number of G, is smaller or equal to I(G) and such that I(RP[n]) is exactly the independence number of this graph.*

Of course one can answer this question as follows: If G is isomorphic to RP[n] we can put I(G) to be [n+1/2] and otherwise let I(G) be zero.

(*) This solution though formally acceptable is undesirable because one purpose of interpolation problems is to include solutions in Graffiti as new invariants, [ACC].

An informal criterion, which can be added to interpolation problems is that the invariant should be of some interest for its own sake. A motivation of Invariant Interpolation Problems is explained in [ACC] (where I moved most of the previous comments to this conjecture.)

To define the solution which I found let us call a vertex v a ***divisor*** *of u, if every neighbor of u is a neighbor of v. The relation* $v \preceq u$ *iff v is a divisor of u is a quasi-ordering of V - the set of vertices of G. A subset F of V is a called a* ***filter****, if for every element v of F, and u, s.t* $v \preceq u$*, u also belongs to F.*

If F is a ***principal*** *filter of V, i.e, if F is of the form* $(v) = \{u{:}v \preceq u\}$ *for some element v of V, then F is an independent subset of G.*

Indeed, let a,b be elements of F = (v). Neither a nor b are adjacent to v, because $v \preceq a$ *would imply that v is adjacent to itself. Now, if b were adjacent to a,* $v \preceq a$ *would imply that v is adjacent a. Hence a,b and thus also F are independent sets.*

If we put now I(G) to be the number of elements of a maximum principal filter of V, then I satisfies all conditions of the interpolation problem.

[V] Siemion Fajtlowicz, On Conjectures of Graffiti, Part V. Graph Theory, Combinatorics and Applications, 7th Quadrennial Conference on Theory and Applications of Graphs. vol 1. 367-376.

[ACC], Siemion Fajtlowicz, Conjectures about Self and Acceleration of Programs, in preparation.

Below there are a few more invariant interpolation problems for the independence number. Lower interpolation problems are those asking for the lower bounds. Of course one could ask an interpolation problem, for any graph, but below they are selected by Graffiti, because the program "needs" them to accelerate algorithms for the independence number, [ACC].

The next seven questions are

Invariant Interpolation problems

as defined in 814.

815. *Let G(1) be the 3-vertex clique, and let G(k+1) be obtained by amalgamating a vertex of G(1) with the last vertex of G(k). Solve the lower interpolation problem for G(k).*

816. *Solve the lower interpolation problem for generalized Petersen graphs, or even just for the Petersen graph.*

It may seem strange to ask for the interpolation problem for one graph, but the purpose of these problems is to invent new graph-theoretical concepts. See also (*) in 814.

817. *Solve the lower and upper interpolation problem for Cayley graphs over cyclic groups with 3 or even two generators.*

818. *Solve the lower and upper interpolation problem for Paley graphs.*

This problem seems to me hopeless, but once I decided to include interpolation problems it is impossible to omit Paley and random graphs, because they lead the list of problems selected by the program.

These graphs also create an interesting offshoot of the interpolation problems. Essentially we know the independence number of random graphs (and Paley graphs are deterministic models of random graphs). It is 2logn. Hence for the purpose of computing the independence number one could exclude the random graphs, i.e., if one knew how to do this. Perhaps one way to deal with the problem is to propose a sound definition of a "deterministic object." The problem seems easier on algorithmic level rather then formal, because we have various tests of randomness. I do not know however whether there is such a test on the basis of which we could conclude that with the probability close to 1, the independence number is close to 2logn?

819. *Let G_n be the cycle with n vertices, H_n the union of G with itself and and J_n the join of H_n with itself. Solve the upper interpolation problem for J_n.*

820. *Solve the upper interpolation problem for complements of generalized Petersen graphs or just the complement of the Petersen graph.*

821. *Solve the upper interpolation problem for complements of buckyballs or just the complement of the dodecahedron. By buckyball I mean any planar cubic graph in which every face has five or six sides.*

January 96.

Conjectures about Ramseyan Properties

822. *Let P be a class of graphs, and let x and y be two non-adjacent vertices of G = (V,E). We color e = $\{x,y\}$ red if G+e does not belong to P and blue otherwise. Let $R_P(E)$ be the set of all red pairs and $R(G) = R_P(G)$ the graph $(V, R_P(E))$. We define similarly the blue graph $B(G)$.*

*Let us call a class P **Ramseyan** if large enough elements of P have large monochromatic subgraphs. If P is Ramseyan then P(r,b) is the smallest integer n such that every n-vertex graph from P contains a r-element red clique or a b-element* *blue clique. Clearly a class is Ramseyan iff large enough graphs from P have large independent sets. These concepts arose quite naturally in study of cases of equality of conjectures 750 and 753, [DD]. In [DF], we proved that, as conjectured at once by* **DeLaVina** *after these concepts were defined, for the class of graphs of chromatic number k, P(r,b) = 1 + k(r-1)(b-1) and the result is based on the lemma that (with respect to the fixed chromatic number) the graph R(G) is a union of disjoint cliques. It would be nice to find more interesting properties of graphs for which either red or a blue graph has a simple structure. I like to think of red and blue graphs as partial complements.*

Together with **Ermelinda DeLaVina and Paul Erdos**, *we have shown that in the class of triangle free-graphs, P(r,r) is about the order* r^3, *but we do not know much about the* K_4*-free case. We conjecture however that in the* K_n*-free case, the critical graphs are disjoint unions of maximal Ramsey graphs. 1. 96.*

9. 97. The triangle-free case begs for the following generalization: what is the minimum number of elements of a metric space so that for a given $\epsilon > 0$ *and an integer* k *either there are* k *points with all mutual distances* $\leq \epsilon$ *or* k *points with distances strictly greater than* ϵ. *The case of three-valued metric spaces corresponds to Ramsey Theorem for graphs, but I think that the question is of interest for other metric spaces. Actually in this formulation it is even more clear that Ramsey Theorem is generalization of the pigeon-hole principle.*

The simplest class of metric spaces for which this problem may be of interest are (geometric) lattices. The independence in these graphs was given a lot attention because it is the sphere packing problem. I do not know however if there are any Ramsey-like results in geometry of numbers.

Once the subject are metric spaces, it is of interest to ask a for a continuous generalization, so I asked **Fred Galvin** *the following question:*

" Let M be a metric space of cardinality m. Let's say that M is Ramseyan if for all sufficiently small $\epsilon > 0$, M has either subset S of cardinality m such that every two points in S are further away than ϵ, or subset of cardinality m in which every two points are closer than ϵ."

Within a day or two Fred sent the proof that M has the required property if m is regular and doesn't otherwise and then he settled the question completely:

"I. If m is a regular cardinal, then every metric space of cardinality m is Ramseyan.

We may of course assume $m > \omega$. Let m be an uncountable regular cardinal, and suppose X is a non-Ramseyan metric space of cardinality m. For each positive integer k, let Y_k be a maximal subset of X such that any two points in Y_k are farther away than 1/k, and let Y be the union of the Y_k; so Y is a dense subset of X. Since X is non-Ramseyan, each Y_k has cardinality $< m$. Since m is regular and uncountable, $|Y| = n < m$. Since X is a metric space, and has a dense subset of cardinality n, it follows that every open cover of X has a subcover of cardinality at most n. Since X is non-Ramseyan, each point has a neighborhood of cardinality less than m. Taking a subcover of cardinality at most n, we get a contradiction to the regularity of m.

II. If m is a singular cardinal of cofinality ω, then there is a non-Ramseyan metric space of cardinality m.

Proof. Let $m = m_1 + m_2 + \ldots$. Let X be the union of disjoint sets X_i with $| X_i | = m_i$. For points x, y in X, x not $= y$, define d(x,y) = 1/i if x, y in X_i; d(x,y) = 1 if x in X_i, y j, x not $= y$.

III. If m is a singular cardinal and $m \leq 2^\omega$, then this a non-Ramseyan metric space of cardinality m.

Proof. Let X be a discrete sum of fewer than m separable metric spaces, each of cardinality ¡ m.

Later Fred sent a complete solution but I have problem with reproducing it here, since my TEX abilities are well below inaccessible alephs. There is some possibility however that it is not just my inaptitude. I remember that Professor Knuth remarked once that his mathematical interest and one of his colleagues meet exactly in the class of countable sets.

In any case my TEX may be good enough to state the result:

Theorem. There exists a non-Ramseyan metric space of cardinality m iff m is singular and, either $cf(m) = \omega$, or else $n^\omega \geq m$ for some cardinal $n < m$." 9. 97.

Fred excluded the case of $m = \omega$, but it seems to me that this case gives a somewhat easier to visualize proof of Ramsey Theorem in the following form:

Theorem: Let M be an infinite metric space of cardinality m and let ϵ be a number > 0. Then either M contains a subset X of cardinality m such that the distance between any two points of X is $\leq \epsilon$ or an infinite subset Y such that the distance between any two points of Y is $> \epsilon$.

Proof: Let Y be a maximal set such that the distance between any two of its points is $> \epsilon$. If Y is finite then the ball B of radius ϵ around one of its elements y has cardinality m. Repeating the operation for $B \setminus y$ we can construct a set of points of cardinality m whose mutual distances are $\leq \epsilon$. For $m > \aleph_0$ we use AC in the form $m^2 = m$. To get Ramsey Theorem for graphs we put $d(x, y) = 1$ iff x is adjacent and different from y and d(x,y) = 2 - otherwise. 9. 97.

12. 97. Soon after I presented this proof in my graph theory class, and I have impression that students found it much easier grasp than the usually given proof." 12. 97.

10. 96. For several months I forget to mention that there are two kinds of numbers P(r,b) of interest. One as defined above and the other is the smallest integer n such that every graph with n **or more** *vertices contains an r-element red clique or a b-element blue clique. It is easy to give examples of Ramseyan properties for which these two numbers are different, but I do not know any examples of hereditary properties (if G belongs to P then every induced subgraph belongs to P for which these two numbers are different.) The version defined here is of course more useful, but the point is that the other one is easier to work with. It would be nice to have general sufficient conditions under which both numbers are equal. 10. 96.*

Bela Bollobas and Oliver Riordan, Cambridge and Memphis University *told me that they proved that if P is the class of* C_4*-free graphs then there is no upper bound of the form* n^k *for* $P(n,n)$*. 2 .96.*

Soon after that they disproved our above conjecture for K_p*-free graphs with p* $\geq$ *4, by a very sophisticated probabilistic argument, [BR1]. Our conjecture however may be still true for p = 3.*

Bela Bollobas conjectures that if P is a downward monotone property of graphs (i.e., if G ϵ *P then so does every subgraph of G) then either R(G) or B(G) contains an independent sets of size logn. He stresses that he hardly knows any examples at this point. 2. 96.*

In conjectures below the red and blue graphs are taken with respect to a fixed chromatic number.

Let $c(v)$ be the chromatic number of the set of neighbors of vertex v and let c^* be the number of distinct $c(v)$'s.

Conjecture: $2c^*$ is not more than the number of distinct components of $R(G)$.

If $R(E)$ is the identity relation then the conjecture asserts that c^* is not more than $n/2$.

If true this conjecture is the best possible, because of the following example: Let $K + D$ be the union of a clique and an empty graph of the same order: joining vertices of D to subsets of K of different cardinalities we get $c^* = n/2$. In view of this example it is of interest how large can be c^* in the class of K_n −free graphs. **Paul Erdos and Siemion Fajtlowicz**, *2.96.*

Michael Molloy, University of Toronto *proved that c^* is not more than $n/2$ and characterized graphs critical with respect to c^*. Some of his lemmas are of independent interest, particularly one which is a chromatic variation of Tutte's Theorem. 11. 96.*

Molloy's proof is based on Berge's variation of Tutte's Theorem which is a lower bound for the number of odd components in the complement of $G - X$. The idea is related to 595. It is of interest what is the lower bound for the number of (not necessarily odd) components, which is enough to prove the theorem. Molloy also remarks that he settled the conjecture in the case when the red graph has no edges. Perhaps one can deduce from [DF] the general case. 9. 97.

[BR] Bela Bollobas and Oliver Riordan,

On some conjectures of Graffiti, preprint 96.

[BR1] Bela Bollobas and Oliver Riordan, preprint 96.

[DD] Ermelinda DeLaVina, PhD dissertation, UH 1997.

[DF] Ermelinda DeLaVina and Siemion Fajtlowicz, Ramseyan Properties of Graphs, Electronic Journal of Combinatorics 96.

[M] Michael Molloy, Chromatic Neighborhoods, preprint '96.

823. *Let e(v) be the number of vertices at even distance from v, and R(G) the graph defined in 822. Then the order of the smallest component of R(G) is not more than minimum of e(v).*

Bela Bollobas and Oliver Riordan and a referee of [DF] *showed that this conjecture is false.*

824. *The chromatic number of G is not more than its clique number plus the number of isolated vertices of R(G).*

Bela Bollobas and Oliver Riordan and a referee of [DF] *showed that this conjecture is false for every chromatic number* $k \geq 3$.

825. *The chromatic number is not more than the number of non-positive eigenvalues of the complement of the blue graph.*

826. *Let c(v) and e(v) be the functions defined in 822 and 823. Then chromatic number of G is not more than maximum of c plus minimum of e.*

Let $r = \max c$. **Bela Bollobas and Oliver Riordan** *showed that this conjecture is false for every* $r \geq 2$, *and they proved that for* $r = 1$, *i.e., for triangle-free graphs, the conjecture is correct with a similar result for the odd version.*

827. *Chromatic number of G is not more than its average dual degree plus the number of positive eigenvalues. (256.)*

828. *The minimum degree of B(G) is not more than the length of the longest path of the complement of G.*

Proved by **Bela Bollobas and Oliver Riordan**.

February 96.

829. *the maximal frequency of c defined in 822 is $\geq$ the number of components of $B(G)$.*

830. *residue of the blue graph is $\leq$ maximum eigenvalue of Laplacian of G.*

831. *residue of the blue graph is $\leq$ 1 +max degree of the $R(G)$ + the average degree o G.*

832. *The average distance of G is $\leq$ 1 + minimum degree of $B(G)$ + the average temperature of complement of G.*

833. *residue of $B(G)$ is $\leq$ residue of complement of G.*

834. *the average distance is $\leq$ 1 + maximum temperature of complement of G.*

This conjecture has nothing to do with red or blue graphs, but it was made by this version, and it is closely related to 127.

Gilles Caporossi *and* **Pierre Hansen, University of Montreal** *note that two triangles joined by a 7-vertex path are counterexamples 3.97. Attaching two vertices of degree k to two antipodal k-vertex paths of a large cycle one can get 2-connected counter-examples, but are there counterexamples of higher connectivity? This conjecture was refuted with aid of AutoGraphiX, comp conj. 3, 62, and 127. 6. 97.*

March 96

Conjectures 835 - 839 were generated by Ermelinda DeLaVina. The conjectures are about triangle-free graphs, and the red and blue graphs are defined with respect to this property, (see 822.) In particular a pair of vertices is red if they are at distance 2, and blue, if their distance is at least 3, or they are in different components. Graphs in these conjectures may be disconnected. The conjectures were tested against about 80 graphs.

835. *the red clique number* $\leq$ *1 + the independence number*

− counter-independence of the complement of the blue graph.

DeLaVina *found three counter-examples to this conjecture the first two of which turned out to be isomorphic to the Heawood graph and the (4,6)-cage. In all of her examples the difference between both sides of inequality is 1, and she asks if somewhat weaker conjecture is true, for example with 2 instead of 1 on the right side of inequality. 5. 96.*

comp 838.

836. *the red clique number* $\leq$ *(1 + blue jet number)* $\times$ *residue of the blue graph.*

837. *the red clique number* $\leq$ *maximum red degree + reciprocal of counter-independence number of the complement of the red graph.*

The conjecture is correct. If the clique number is more than maximum degree then graph has a component which is a maximal clique. In this case, the complement is an empty graph or a join of a maximum independent set with another graph and in either case the counter-independence number is 1. s.f. 4.96.

838. *the red clique number* $\leq$ *blue residue + maximum deficiency.*

The deficiency of a vertex is the number of non-edges in the set of its neighbors.

If G is the incidence graph of a projective geometry of order p, then the G is regular of degree d = p +1, and the deficiency is $(p+1)p/2$. *The red clique number is however* $d^2 - d + 1$, *and the blue residue is small because the blue degree is close to n/2. s.f. 5. 96.*

DeLaVina's examples in 835 also realize the bound $d^2 - d + 1$, *which is the best possible, and projective geometry graphs are also counterexamples to 835, but not to the question DeLaVina posed there.*

Let P(G) be the set system obtained from a triangle-free graph G by associating with every vertex the set of its neighbors in a maximum red clique P. Let us call this set **lines** *of P(G) and denote it by L, i.e., P(G) = (P, L).* **DeLaVina** *proved that if the red clique number is equal to* $d^2 - d + 1$ *then G is the incidence graph of a projective geometry. 5. 96.*

It would be interesting to characterize those graphs in which the red clique number is $d^2 - d$*. Perhaps this time, one have to make an assumption that graph is transitive to get an equally neat description. An example is a 12-vertex Franklin graph which is bipartite, has diameter*2*, but still its red clique number as well as the blue independence number is equal to the independence number. DeLaVina found the following representation of this graph: Take two copies of* C_6 *join corresponding even-numbered vertices and then odd corresponding by counterclockwise shift by 2.*

839. *the red clique number* $\leq$ *number of blue isolated vertices + maximum of odd vertices.*

March 96

Fullerenes

840. *A* **fullerene** *is a cubic planar graph in which every face has five or six sides. A fullerene is called an IP isomer if it contains no adjacent pentagons. It is thought that chemically stable fullerenes must be IP isomers, [FM]. I am indebted to Darko Babic for sending me 10 examples of fullerenes which are considered to be the most stable, 9 examples which are thought to be very unstable, as well as providing quite a few quick and very useful tutorials to introduce me to the subject.*

Graffiti made number of conjectures for stable fullerenes which are false for each of 9 nonstable examples, but I think that at least some of these conjectures should be valid for any IP isomer, so before I'll get other examples I am not going *to include them here. Below is the very first conjecture made by the program about fullerenes:*

The independence number of stable fullerenes is n/2 -6.

At the moment I do not even know whether the program did not undercompute the independence number, so perhaps the conjecture should be interpreted as the statement that the independence number in stable fullerenes is at least n/2 -6. In any case I can show that the independence number of any IP isomer is at most n/2 -4 and that for any fullerene the independence number is at most n/2 - 2. The last inequality is very easy to show, and the only example of a fullerene with equality, of which I know, is the dodecahedron, but I think that there should be other such examples. In all of them (if there are such examples,) the pentagons can be paired so that they share a side. If an IP isomer contains no hexagons H with two pentagons sharing a pair of opposite sides of H, then the independence number is at most n/2 -6.

7. 98. **Patrick Fowler** *and* **Kevin Rogers**, **Exeter University** *have found that for every* n *between 26 and 70 there is at at least one* n*-vertex fullerene with the independence number* $n/2-2$. *More interesting, these fullerenes appear in the contex of boron-nitrogen derivatives. See the text below with the date of Fourth of July. 7. 98.*

One of the stable examples has independence number equal to $n/2-6$. C_{60} *is covered by disjoint pentagons so its independence number is at most 24. Gordon Royle has verified (with a program) that the 70 vertex example also has the independence number equal n/2-6, but that still leaves 7 examples which may have the independence number n/2-5 or n/2-4.*

Darko Babic, Rugjer Boskovic Institute, Zagreb, Croatia *found a number of IP isomers which have independence number at least n/2 -5, and later he has verified that the same is true in two of his 10 stable examples, 4. 96.*

I think that the minimum number of edges, as well as the

(*) minimum number of vertices which have to be deleted to make the graph of a fullerene bipartite, should be related to its stability,

and this invariant is definitely related to this conjecture, comp also 845, as well as 753 and 750 (*). The last bound is one of the only two (known to me) lower bounds for the independence number in graphs, which is of any use in estimation of the independence number of fullerenes. The other is a result of Mark Ramras about graphs which can be covered by disjoint pentagons and which implies that the independence number of the stable C_{60} *is* $2n/5 = n/2 - 6 = 24$*. Ramras' result has very strong assumptions, but perhaps they can be weakened.*

Fullerenes may be good test problems for independence algorithms. The problem is NP-hard already in the planar cubic case, and the larger the independence the more difficult these problems seem to be (I mean that it is the size of the independent set, not the number of vertices is the most significant factor.) On the other hand fullerenes are "almost bipartite" so they present opportunity for generalization of algorithms for independence in bipartite graphs.

Let S_n *be the graph obtained from two parallel n-cycles placed horizontally on the boundary of a cylinder by first joining the corresponding vertices by paths of length 4 and then making the graph cubic by joining each of the upper path vertices to the lower ones on the next path in say, counterclockwise direction.*

We ran into these graphs independently with Staton years ago when we tried to construct cubic triangle-free graphs with low independence numbers. S_n*'s are* *close relatives of fullerenes; S_5 is the dodecahedron, and S_6 is the unique 24-vertex fullerene.*

10. 96. For odd n, these graphs are oddettes defined in 843 10. 96.

If we place one of the S_n's on the top of another and rotate it to make the graph cubic then the result is again a fullerene, for $n = 5$ or 6. and the operation can be repeated producing a natural embedding of these graphs into a torus, by eventually applying the operation to the top and bottom cycles. Let $S_{n,k}$ denote the result of application of this operation k times. This time n can be arbitrary. There are nice pictures of $S_{n,k}$'s described in somewhat different manner) in [BLKB].

We can also consider a Mobious version of S_n's: Let us draw S_n on a long rectangle, delete the leftmost, third from the top vertex, and the second, rightmost. Then twisting and linking by edges the facing vertices on sides of the twisted rectangle we get an embedding of a cubic graph into the Mobious strip (the edges on vertical sides are also deleted). The graph has a hexagon sharing a side with C_4. Creating and joining by an edge the two new vertices on opposite sides of the hexagon one of which is shared with a 4-cycle, we replace hexagon and C_4 by three pentagons. Let us call these graphs M_n.

I wonder what may be mathematical principles which make some cubic graphs unfeasible as carbon molecules apart from bonds being represented as intervals of fixed length (and I think there are also some angle constraints) For example, K_4 and dodecahedron are unstable, but they are feasible.

7. 96. Patrick Fowler told me that angles should ideally be about 120 - 120 - 120 or about 109.47 - 109.47 - 109.47. Discrepancy increases the energy of a molecule making it less stable. 7. 96.

Surely there should be small nonplanar graphs which come close to meeting these constraints, and in any case one can ask what is the minimum angle-discrepancy over all possible 3-dimensional representation of graph with a fixed edge-length. Perhaps for the Franklin's graph the angle discrepancy will be not too difficult to compute taking into account the representation of **DeLaVina** *in 838. 7. 96.*

11. 96. On the second thought, if one could find small examples, they would be likely to occur in nature, so the question is really is what are the smallest examples meeting above constraints. 11. 96.

7. 98. On the third thought I do not understand why I did not mention here the Heawood graph (though I remember I thought about it.) Heawood graph has is bipartite - thus avoiding troublesome pentagons and has girth 6. 7. 98.

I asked about the feasibility of M_n, in view of the known fact that monocyclic molecules seem feasible in the range of up to n =25, [FM], p. 3, and that is what Gunnar Brinkmann, who just attended Exter conference wrote:

" In his talk on the conference E. C. Kirby said that for small numbers of atoms toroidal fullerenes are physically more probable if one includes heptagons and pentagons, because of less stress on the bonds. This was because of some energy calculations carried out on special toroidal graphs with pentagons and heptagons. They don't know whether these graphs are best possible "

Gunnar, 4. 96

In a previous letter Gunnar mentioned that according to the same lecture if every face has 6 sides then the toroidal molecules are unlikely to be stable below 300 atoms.

If M_n's will prove feasible, one could also ask about a feasibility of non orientable molecules which can be obtained by pasting the long cycle of M_n over a face *of $S_{n,k}$, (or perhaps some other n-faced, girth 5 cubic graphs) in the same manner as unorientable surfaces are obtained from sphere with holes covered by Mobious strips. If one thinks of such a molecule as a surface then this is impossible, but since a molecule is a configuration of atoms, perhaps such arrangements are feasible, though they sound far-fetched at the moment. 4. 96.*

8. 31. 96. **Norma DeLaVina** *built out of hand a model of Petersen graph out of toothpicks and styrofoam balls which I gave to Ermelinda as a birthday present - angles of course were a disaster, but I am glad that I did not mention this condition to her, because after the fact, it occurred to me that it is of interest to consider just equi-angular models. Sticks "intersect" in Norma's solution, and I am not sure if it is possible to avoid this preserving equality of the bonding sticks. Is her solution unique? Also, if anybody rushed already opinion about my birthday gifts, I threw in a soccer ball too. 8. 31. 96.*

I ran on the web into a number of articles about nanotubes - fullerene-like structures - which admit pentagon-heptagons pairs of faces. I also found references to helically coiled carbon nanotubes (there are also metallic and even organic (viral) versions of fullerenes) which I found intriguing, because I thought that M_n's could lead to hellical structures. Intuitively, it seemed to me that for large enough n M_n's should be feasible, though unstable. Nevertheless if we form $M_{n,k}$'s from M_n's similarly as $S_{n,k}$ is obtained from S_n (by attaching along the long cycle another copy of itself) then perhaps these structures can be even stable. From the technological point of view these molecules would have interesting property that they could be replicated not only by gluing, but also by cutting (which was the reason I thought of hellical structures) producing linked and doubly twisted molecules. I can't imagine (though *it does not prove much) any principles preventing linked loops of say monocycles. 7. 96.*

7. 98. Such molecules are indeed feasible, as I found during the talk of Professor Mislow - Molecular Rubber Gloves - at the March DIMACS meeting. 7. 98.

11. 96. *An* **aligning** *is a system* (G, D) *where* G *is a graph and* D *is a graph whose vertex set is* $E(G)$ *- the set of edges of* G*. Two edges of* G *adjacent in* D *will be called* **aligned***. Let* l *be the number of elements of* $E(G)$*. We will be primarily interested in examples of* D *being matchings with* $l/2$ *edges, i.e.,* D *being a set of (disjoint) pairs of edges of* G*. Such systems will be called* **perfect alignings** *of* G*. The* **dual aligning** *of* (G, D) *is the system* (L, ID) *where* M *is a graph whose vertex set consist of vertices and edges of* G*, two being adjacent iff*

(a) v *is vertex of* G *and* u *is an edge of* G *incident with* v*, or*

(b) u *and* v *are two edges of* G *adjacent in* D*.*

and ID *is an independent subset of* M *such that*

(c) every vertex of M *is adjacent to exactly two vertices of* ID*.*

Note that if G *is a cubic graph with a perfect aligning* D *then the dual aligning of* (G, D) *is also cubic, and I will show below that* C_{60} *is a dual aligning of a truncated cube. Morever [F] contains a proof that fullerenes which can be represented as amalgamats of alignings can have at most 60 vertices and in particular* C_{60} *is the only IPR of this form. At this level, including the proof of [F] one does not have to go beyond perfect alignings, but they appear very naturaly in this subject, and the construction of dual alignings in the more general settings occured to me in 788, so it may be relevant to Ramsey graphs.*

Duals of perfect alignings are sometimes easy to visualize: we can place an extra vertex on each of the edges of G *and then join two middle vertices iff they belong* *to aligned edges. I will use the notation* $(G, D) = (G, \updownarrow)$ *where* $\updownarrow$ *is the adjacency relation of* D*. I will also informally identify the dual alignings with the their graphs, particularly when it is obvious what is the independent set* ID*.*

Examples:

1. Let $\updownarrow$ *be the aligning which matches disjoint edges of* K_4 *- the complete graph on 4 vertices. Then the dual aligning of* $(K_4, \updownarrow)$*, is Petersen graph.*

2. Let G be the cube, and let $\updownarrow$ *(again the adjacency relation of D) be its perfect aligning which associate with every edge a non-incident edge belonging to the same face in such a way that each face contains exactly two aligned edges. Such aligning is unique up to the cube symmetries, and it will be called the* **standard aligning** *of the cube. The amalgamat of this aligning is the dodecahedron.*

3. Let G be the graph obtained from the cube by truncating its vertices. Let $\updownarrow$ *be the extension of the standard aligning of the cube to G in such way that the resulting dual aligning is* $G^{\updownarrow}$ *is planar. Then* $G^{\updownarrow}$ *is Buckminsterfullerene. The simplest way to see it, is that* $G^{\updownarrow}$ *is the unique planar cubic planar graph covered by 12 disjoint pentagons in which every face is a pentagon or hexagon.*

*Cubic dual alignings will be called (***pentagonal** *systems (or graphs) We have a natural correspondence between perfect alignings of cubic graphs and pentagonal graphs, and this is the basis of characterization of* C_{60}*.*

Note that if (G, I) is pentagonal graph then the number of elements of I is $2n/5$*, where* n *is the number of vertices of* G*. Thus, in the case of* C_{60}*, I has 24 vertices and thus* I *is a maximum independent set of* I*. The pentagonal structure of* C_{60} *is easy to visualize, (particularly if one has an appropriately colored soccer ball, which I just got at Toys R Us.) The 24 independent vertices must be contained in 8 isolated hexagonal faces which will be called* **octave.** *This structure is a natural* *consequence of a simple counting argument from [F]. Centers of hexagons of the octave form the cube which is the basis of our construction.*

Without the benefit of Toys R Us, the pentagonal structure of Buckminsterfullere may be found as follows. Let us start with a 3-element independent set I in a hexagon H of C_{60}. Let e be an edge emanating from a vertex of H not in I. The other endpoint of e belongs to a hexagon of the octad extending (H, I) and tells us which of two 3-element independent sets extend I.

Toys R Us did not provide the answer to my question whether the number of molecules of C_{60} per a unit of volume may be greater than one might expect from the densest sphere packing in R^3. The density of such packings is known up to about 0.0379 *percent, and according to Rogers, see [CS], "many mathematicians believe, and all physicists know" that the optimal solution to the problem is the familiar fruit-stands packing with the density of 0.7405.. . This packing is also found in crystalline argon. If one could realize the denser packing of Buckminsterfullerene it might be considered as indication that the truncated cube representation of the molecule has a physical manifestation. A denser packing does not have to be found exclusively in C_{60}. Clathrin is a C_{60}-like protein involved in communication between cells.*

The 24-vertex independent sets of C_{60} occur naturally in chemistry, see Taylor, the chapter on halogenofullerenes. Bromine atoms react with C_{60} and since they are too large to attach themselves to neighboring carbon atoms they determine in C_{60} an independent set of vertices. C_{60} has other (up to the graph automorphisms) 24-element independent sets, but those which define $C_{60}Br_{24}$ molecules are exactly the same, as those present in the above construction of C_{60}, i.e., they arise via the correspondence between the pentagonal and aligning systems from the standard aligning *of the truncated cube. Roger Taylor and his colleague John Holloway have also made* $C_{60}Cl_{24}$*, but "although it is not possible to obtain the X-ray structure, it is virtually certain that it will correrspond to that of* $C_{60}Br_{24}$ *(personal communication).*

An amusing incident took place, in connection with the identity of the set. Initially, I copied incorrectly the independent set of $C_{60}Br_{24}$ *from [T], and I was close to challenging chemists opinion (and the patiance of Patrick Fowler and Roger Taylor - though they passed this test with flying colors) that they have the right set. The amusing part, is that I strongly disagree with certain aspects of Platonism (creation vs. discovery) Nevertheless, as documented above, I clearly agree with Diedonne who said - "six days a week, we are all Platonists."*

If one will place large round paper dots on the vertices of the 24-vertex pentagonal set one can see two prominent ortogonal pairs of tropics (bromine alleys) running around the soccer ball. Each tropic has 8 vertices and above it there is a polar square. Tropics are not circles, because every fourth pair of bromine atoms marching along the bromine alleys resides in the pentagon, and the remaining pairs are antipodes of hexagons. The (sphere coding) problem of placing 24 vertices on the surface of the sphere to maximize angle separation distances is solved, [CS], but I have no idea whether bromine atoms form a solution to this problem.

I have mentioned above that the dual alignings came up in 788, but they are results of putting coordinates of independent sets in Graffiti as invariants. The coordinate of a vertex v *with respect to the independent set* ID *is the number of neighbors of* v *in* ID*. Let* α_k *be the number of vertices whose coordinate is* k*. I called the* **dimension** *of a graph to be the largest* k *for which* α_k *is not zero.* α_k*'s.*

Alexander, Hopkins and Staton showed that cubic triangle-free graphs of dimension 2 have $n = 10k$ *for some* k*, have girth at least 5, the independence number* *exactly (2/5)n, and have every other vertex adjacent to exactly 2 vertices of I. The generalized Petersen Graphs P(5k,2) are examples.*

In problems which came up with characterization of C_{60} *of interest are are maximal rather then maximum independence sets, but Petersen graphs seem to be closely related to these problems. The Petersen graph was the graph theory graph bc (before* C_{60}*), and the 14-vertex generalized Petersen graphs, is one of the two graphs which show that Staton's 5/14 Theorem is the best possible. The concept of dimension came up because to prove Bill's theorem it is enough to show that if we maximize* α_3 *then* $\alpha_1 \leq 2.5\alpha_3$*.*

Of definite interest seem now maximal independent set which minimize the number of nonzero α_k*'s, and particularly those for which this number is 1. All platonic graphs have such sets. Perhaps one can characterize all transitive graphs with this property. In general, of interest is the following structure. A* **designed graph** *is system consisting of a graph and a partition of vertices such that every vertex have the same number of neighbors in each part; examples corresponding to bipartite graphs can be formed from designs.*

Like here the numbers 8 and 24 pop up all the time in sphere packing problems. Probably the link is the Mathieu group M_{24} *whose one of three generators described in [H] is involution, and this involution is probably related to the standard aligning of edges of the truncated cube, or (and) bromine alleys, but I do not know how the two other generators should be represented on the pentagonal set of* C_{60} *in a way which would tell us something interesting about the molecule. In Passman's description involutions play even more prominent role, and I think that an expert will see at once the right action of* M_{24}*, unless bromine alleys do not exist in Platonic world. Involutions played a role in discovery of sporadic simple groups emphasized* *by Aschbacher. iT would be interesting to know whether aligned graphs have place in this picture. 11. 96.*

On March '98 DIMACS meeting Patrick Fowler suggested to describe all 24-element independent sets of $C_{60}Br_{24}$. *Two such sets are (chemically) the same if the graphs induced by their complements are isomorphic. Apart from being independent bromine atoms must obey the* **parity rule** *that each component of the complement of the independent set must must have even number of vertices.*

The problem was solved with a computer by the first 3 authors, and then I contributed somewhat toward more direct descriptions of these sets, which give an intriguing example in the independence theory. Each of them can be obtained by rotating some of the triples of independent sets residing in a face of an octet of C_{60}, *[F]. The octet of* C_{60} *can be thought as the pentagonal independent set of the dodecahedron of hexagonal faces of* C_{60} *(the subgraph of the dual induced by vertices of degree 6. The octet has the structure of the cube which is the basis of the representation of* C_{60} *in [F], and last, but not least, the rotated faces of the octet correspond uniquely to independent sets of the cube.*

There is another independence-related occurance in this paper. The complement of the bromine set is a maximal independent set in the line graph of the dual of C_{60}.

Apart from the 24-bromomofullerene there are only two other well characterized examples corresponding to $n = 6$ *and* 8*,[AFS], and* $C_{70}Cl_{10}$ *is discussed in [AFS]. It is interesting that in the first one the bromine atoms are not independent. Actually it occured to me that perhaps this can occur with more than 24 bromines, if some of them are inside the shell and Patrick wrote that this possiblity was considered by chemists:*

"The possibility of attachment inside the cage has been mentioned for H and F atoms (much smaller that Br) but the question is not so much whether if they were placed inside it would be a low energy configuration, but how they could get there in the first place. The chemical process of bromination is fairly gentle and starts from an intact cage. To get Br through the skin of the cage would need large energies to braek the cage edges, and is very unlikely. The only certain endohedral atoms produced in experiment so far are He, which is small and probably slips through a face in a high-energy collision (the experiment that makes it uses very high presuure and a 'cooking' process), and some metal atoms, which probably open up the cage by a more complicated chemical reaction not available to Br."

Patrick

Since independence of bromine atoms is the the result of steric strain, therefore we still have a purely geometric problem whether $n \geq 24$ *is possible. It is interesting though that David Jones (undoubtedly a close, and bolder relative of Dr. Indiana Jones) who first conjured up the idea of carbon cages, thought that the stability would require several hundreds of atoms, and this could make inside-outside combination more likely. That of course still would depend on the process of formation of fullerenes - which was one of motivations for the "spiral conjecture," [FM]. It is interesting that the steric strain is discussed in the "Atlas of Fullerenes" in terms of purely combinatorial indices.*

The parity rule is a weak mathematical form of the rule that "each separate π *system in the addition pattern should have a closed* π *shell, (i.e filled molecular orbitals), [FHR]. A stronger, but more difficult to use is the* **parity condition:** *the number of positive and negative eigenvalues is the same. Graphs with this properties came up on occaison of conjectures 6. 98.*

Patrick Fowler suggested, and began computer search, for 29 or 30 element bromine sets of C_{70} *containing edges. Assuming that the subgraph induced by bromines consists of isolated atoms and edges, I can show that the number of needed edges is at least 3 in the former and at least* 5 *in the latter case:*

Let B *be the* $b-$ *element set of bromines, let be* a_k *be the number of not brominated carbon atoms adjacent to* k *elements of* B*, and let* e *be the number of edges of* B*. For arbitrary fullerene with* n *atoms we have:*

$$a_1 + a_2 = n - b$$

and

$$a_1 + 2a_2 = 3b - 2e$$

.

Indeed, the parity rule implies that $a_3 = 0$ *and the second identity follows from counting edges emanating from* B*; I assume here that they are isolated, but if the condition will prove chemically not viable, one can still get somewhat weaker bounds.*

$$a_1 = 2n - 5b - 2e$$

and

$$a_2 = 4b - 2e - n$$

Thus

$$4b - 2e - n \leq n - b$$

i.e

$$b \leq 2n/5 + 2e/5$$

which proves the claim for C_{70}. *7. 98.*

Fourth of July, 98

From Patrick Fowler

There are 159 fullerenes between 20 and 70 vertices (inclusive) that meet that equality in the stability number ($\alpha = n/2 - 2$, *s.f.) There is at least one at every vertex count except 22 (no fullerene) and 24 (the stability number of the unique 24 vertex fullerene is 9). The pentagon arrangements in all of these can be broken down into fusions of pairs, though of course it is not necessary for all pentagons to occur in isolated pairs (the first fullerene with six isolated pentagons pairs - the first IPP fullerene - is at 50 vertices, two more vertices than might be expected from the Euler formula).*

These cages have already turned up in a theoretical investigation of boron nitride analogues of the fullerenes (Fowler, K M Rogers and G Seifert, in progress) and solve the following problem: to decorate the fullerene with B and N atoms such that no B B pair is adjacent, all hexagonal rings are BNBNBN alternating and the number of N N contacts is minimal (ie 6). The decoration has formula B_xN_{x+4} *and maps to the stability problem with B = in, N = out of the stable set. Some thoughts on the IPP subset of these cages is given in P.W. Fowler, Phil. Trans. Roy. Soc. Lond. A 343 (1993) 39-52. Also issued as a chapter in Fullerenes Eds. H.W. Kroto and D.R.M. Walton, Cambridge University Press, 1993, 39-52 Systematics of fullerenes and related clusters where the cages that we want here are 'even' cages in the terminology of that paper, and the roles of B and N have now been reversed.*

The fullerenes with maximum independence provide interesting examples. Let us call planar graph G **independence-complete** *if G has an independent set inducing maximum independent set all faces of G. Since fullerenes have 12 pentagonal faces $3\alpha \leq 24 + 3(n/2 - 12)$ and thus they are independence-complete if and only if their independence number is $n/2 - 2$. This provides for a very efficient algorithm to decide whether a fullerene is independence-complete: start with one of 3-element independent sets of a fixed hexagonal face and continue to extend the independent set to adjacent hexagons as long as possible. Excluding cases of hexagonal islands, we will cover all hexagons and then we can verify, whether it can be (unless it was already done) extended to 12 pentagons, so that each of them contains a 2-element independent set. The procedure may have to be repeated to start with the other 3-element independent set of the initial hexagon and perhaps it will have to be run a few times in the presence of hexagonal islands, but the algorithm is linear assuming the right representation of data. Of course, for large n there is one dominating island, so algorithmically it is not a problem.*

Most of the time, in the case of independence-complete fullerenes, the choice of the initial hexagon and its independent set won't matter, because pentagonal islands must have even peripheries. We can use the same procedure starting from the boundary of a pentagonal cluster and then the procedure will go even faster because most of the time the right (if there is one at all) independent set restricted to a pentagonal cluster is unique. An exception is a cluster consiting of 5 pentagons surrounding the sixth one.

Let G be now an arbitrary graph and let D be a "generic" family of subsets of G (hopefully it will be clear later what I mean by this, but the example in which $D(G)$ is *the family of maximum independent set in independence-complete fullerenes may be already of sufficient interest.) A* **picture** *of G is a collection of* **snapshots** (X, D), *where X is a set of vertices of G called a* **face** *and D is an element of $D(X)$, called a* **decoration**. *Suppose now that all faces have the decorations of the same size, and the same is true for all non-empty intersection of every two faces. For every set of vertices of G which can be represented as a union of faces we define the rank function r by the rule $r(X \cup Y) + r(X \cap Y) = r(X) + r(Y)$.*

The idea is that we have a matroid-like structures with the rank function defined on certain subsets of G: those which can be represented as unions of faces. To get somewhere one probably has to impose a condition on the independence number of the intersection of faces.

[AFS] J. S. Austin, P.W. Fowler, J.P.B. Sandall, P.R. Birkett, A.G. Avent, A.D. Darwish, H.W. Kroto, R. Taylor and D.R.M. Walton, A theoretical characterisation of $C_{70}Cl_{10}$: the role of $1,4$ addition across hexagonal rings, J. Chem. Soc. Perkin Trans. 2. 1995, 1027-1028

[AHS] On the independence dimension of a graph, Ch. Alexander, G. Hopkins, and W. Staton, Congressus Numerantium 28(1980),211-215

[ASC] M. Aschbacher, Sporadic Groups, Cambrgidge University Press.

[ASC1] M. Aschbacher, Finite Group Theory, Cambridge Studies in Advanced Mathematics,

[BLKB], Graph Invariants for Fullerenes, Balaban, Liu, Klein, Babic, Schmalz, Seitz and Randic, J. Chem. Inf. and Comp Science, 1995, 35.1

[CS] J. H. Conway and N. J. A. Sloane, Sphere Packings, Lattices and Groups, Springer - Verlag, Second Edition.

[F] Siemion Fajtlowicz, On representation and characterization of C_{60}, in preparation.

[F1] Siemion Fajtlowicz, Program Accelerators, in preparation.

[FHR] P. W. Fowler, P. Hansen, K. M. Rogers and S. Fajtlowicz, $C_{60}Br_{24}$ as a Chemical Illustration of Graph Theoretical Independence, to appear in Perkin Transactions.

[FM] P. W. Fowler and D. E. Manolopoulos, An Atlas of Fullerenes, International Series of Monographs on Chemistry, Clarendon Press, '95.

[H], M. Hall, The Theory of Groups, MacMilan.

[HS] D. A. Holton and J. Sheehan, The Petersen Graph, Australian Mathematical Society, Lecture Series 7.

[MR] Mark Ramras, Maximal Independent Sets in Graphs with Vertex m-covers, Congressus Numerantium, 39 (1983) pp 463 - 472.

[T] Roger Taylor, The Chemistry of Fullerenes, Advanced Series in Fullerenes, vol 4. World Scientific.

April 96

Conjectures below were tested on 121 fullerenes, most of which were generated by Fullgen, a program written and offered to me by Gunnar Brinkmann. This program is invaluable in testing of conjectures, though the largest example which I used so far has 100 vertices, well below the capacity of this program. The program is a real bb gun.

841. The number of positive eigenvalues of any fullerene is greater or equal to the number of its negative eigenvalues.

I think that this conjecture is interesting, because usually it is the other way around. In particular the Cvetkovic's bound for the independence (see conjectures *about plants (which start with 345) is better in the case of nonpositive eigenvalues (and similarly as the lower bounds mentioned in 859, it seems to be the only useful upper bound for the independence in fullerenes.)*

For two of 121 examples we have equality, and interestingly enough both of them are two of Babic's unstable examples, i.e, they are geotropic plants. The number of cases with equality may be larger because the actual independence number may be greater than computed by Graffiti.

This conjecture has interesting stability-sorting pattern (for the definition see 845). For all unstable examples the difference is 6, and for all stable at most 6, with the difference 6 for one of them which notoriously sticks out of the group. The maverick has 78 atoms and the symmetry group D3h.I have no choice, but to conjecture that this example is much less stable than the other nine. I should remind however that stability and instability of both groups is itself a matter of conjectures.

Below, with his kind permission, are fragments from letters of **Darko Babic***:*

I have systematically checked all the isomers I have: these are all isomers with up to 70 vertices and all IPR isomers up to 100 vertices (including the upper bounds). I have computed spectra for all them and saved the inertia (number of positive, zero and negative eigenvalues). After that I searched for extrema and got the following:

24.	2 4 4	26.	0 4 4	28.	1 4 3	30.	0 6 4	32.	1 5 2
34.	0 4 4	36.	2 6 4	38.	2 6 3	40.	1 6 4	42.	2 6 3
44.	2 6 2	46.	0 6 2	48.	2 6 2	50.	1 6 2	52.	1 6 2
54.	0 6 2	56.	2 6 2	58.	1 6 2	60.	2 6 0	62.	1 6 2
64.	1 6 2	66.	0 6 2	68.	2 6 2	70.	1 6 1		

For IPR isomers:

72.	*0 0 0*	*74.*	*0 2 2*	*76.*	*0 4 2*	*78.*	*0 6 0*	*80.*	*0 6 2*
82.	*0 4 2*	*84.*	*3 4 0*	*86.*	*0 6 2*	*88.*	*0 6 2*	*90.*	*0 6 0*
92.	*0 6 2*	*94.*	*1 6 2*	*96.*	*0 6 0*	*98.*	*0 6 2*	*100.*	*1 6 1*

The first column is the maximum number of zero eigenvalues found in any isomer with the number of vertices denoted in the leftmost column (N). The second column is the maximal difference between the number of positive and negative eigenvalues found among the isomers with fixed number of vertices. Similarly, the third is the minimal value of the same difference ...

... The hypothesis version I am now inclined to claims that there are at least n/2-3 positive and at least n/2-3 negative eigenvalues for any fullerene. This formulation was made having in mind the way Manolopoulos, Woodall and Fowler applied for leapfrog fullerenes, but I don't see the convenient n/2-3 vectors needed for application of the Rayleigh theorem.

Darko Babic, 4.96

Patrick W. Fowler, University of Exeter *told me that he knows a fullerene of order 628 in which the number of negative eigenvalues is one more than the number of positive, [FLR]. 4. 96.*

After the message from Professor Fowler it occurred to me that the following is the reason that the difference between the number of positive - $p(G)$, and the negative eigenvalues - $n(G)$ can not be too large, though the size of the difference probably depends on a correct version 843:

for bipartite graphs $p(G) = n(G)$, and by Cauchy's Interlacing Theorem, if G is an induced subgraph of H then $p(G) \leq p(H)$, and $n(G) \leq n(H)$. For large n, one *can show that a n-vertex fullerene has a bipartite subgraph of order $n - clogn$, and hence the difference is at most of this order. I will outline (for now) the proof in 843 (*), but perhaps later I'll include the precise estimate, because it may be of interest what is the constant c.*

This example brings up the following problem. Let us call a one-vertex extension H of G positive if $p(H)$ $p(G)$, and negative if $n(H)$ $n(G)$. What makes an extension positive? Results answering this and similar questions for non-negative eigenvalues, etc could be also very helpful with 446. Perhaps there are theorems of this kind for symmetric matrices. 4. 96.

According to [FLR] the property that the number of positive eigenvalues is more than negative has a clear-cut chemical equivalent. Patrick Fowler notes that while a "mathematician would no doubt agree with Sherlock Holmes that a rule with exception is not rule at all", for a chemist counterexamples in this case are proverbial confirmation of a rule.

I like exceptions, and on this occasion, I side with chemists because the "no exception rule" is undoubtedly invention of Professor Moriarty. In any case, the rule in question is already exceptional, because usually we have more negative eigenvalues and this, I think, makes the situation more interesting.

[FLR] P. W. Fowler, Fullerene Graphs With More Negative Than Positive Eigenvalues; The Exceptions That Prove The Rule of Fullerene Deficiency? J. Chem. Soc. Faraday Trans. 93(1), 1997, pp. 1 -3.

[MWF] D. E. Manolopoulos, D. R. Woodall and P. F. Fowler, Electronic Stability of Fullerenes: Eigenvalue Theorems for Leapfrog Carbon Clusters, J. Chem. Soc. Faraday Trans. 88 (1992) 2427-

842. *Let v be a vertex maximizing the number of v-horizontal edges, and let e be the number of vertices at even distance from v. If G is a fullerene then the independence number of G is at most e -2.*

843. *The independence number of any Fullerene is at least* $n/2 - 8$. *comp 840.*

Let $T(G)$, *where G is planar, be the graph whose vertices are faces of* G, *two being adjacent, iff they share a common vertex. If G is cubic then* $T(G)$ *is its dual, but in general it is the tertiary graph :-).*

*Lemma. (***DeLaVina** *and myself.)*

Let $\omega(G)$ *denote the number of odd faces of a planar graph* G. *The minimum number of vertices which have to be deleted to lower* $\omega(G)$ *is equal the length of a shortest path joining two odd faces in* $T(G)$.

Proof: Let $F_0, ..F_d$, *be a shortest sequence of faces, adjacent in* $T(G)$, *joining two odd faces* F_0 *and* F_d. *If d=1 we have two or more odd faces sharing a vertex v.*

Suppose that v *belongs to m odd faces and m is odd. Let F be the face resulting from deleting* v. *Then F is odd-sided, because deletion of* v *destroys an odd number of odd faces and thus the number of intact odd faces must be also odd.*

To clarify, by an odd face we mean a vertex of odd degree in the dual graph and we assume that every loop contributes 2 *to the degree of a vertex. Then the sum of degrees is twice the number of edges and we can use the parity argument. Since F is an odd face,* $\omega(G)$ *is lowered by at least 2 (if the number of faces containing* v *is odd then the face resulting from deletion of* v *is odd, but* v *is contained in at least 3 faces.) The case of even m is similar, i.e., in both case* ω *is lowered by at least* 2.

The lemma follows now by induction on d; if $d\,1$ *then deleting a vertex in* F_0 *and* F_1 *leaves* ω *unchanged, but shortens the distance between two closest odd faces. 4. 96.*

This proposition gives a lower bound and an algorithm which will be called ***Oddette****, for a large bipartite subgraph in terms of (minimized) sum of pairwise distances between odd faces of fullerenes. Indeed, after removal of the first pair of odd faces, the minimized distances between remaining faces may only decrease, even if the graph is disconnected in the process. The precise description of Oddette is that it first deletes vertex* v_{n-1} *which belong to a maximum number of odd faces.*

I would conjecture now that every planar triangle-free graph of maximum degree 3 can be made bipartite by removal of $n/4+1$ *vertices, (a known conjecture asserts that every such cubic graph has an independent set with 3n/8 vertices). Perhaps even Oddette may accomplish this task. Here is my rationale: cubic graph has n/2 +2 faces, and since each round of Oddette lowers* ω *by 2, (unless the graph is IOF - i.e., all odd faces are isolated) in seemingly the worst case, i.e., when all faces are odd we need at most n/4 +1 rounds. A* **round** *consist of steps needed to lower* ω.

Let beta(G) be the minimum number of vertices which have to be deleted to make G bipartite.

10. 96. It is easy to give examples of planar graphs in which every face is odd (they will be called **oddetttes** *- the term odd graph is already used for a different class) but cubic oddettes are much harder to find. The only examples which I know at the moment are graphs* S_n *with odd n described in 840 and graphs obtained from odd number of* K_4 *minus edge strung into a cycle.*

Oddettes of maximum degree 3, are much easier to obtain reversing the procedure from the second lemma. The procedure defines a sequence of sets of odd integers (odd faces destroyed in the process) and one could ask which of these sequences can be realized by Oddette - the algorithm.

Another related question is: a given pair of integers (d, ω) *what is the number of vertices of a smallest planar graph* G *in which the shortest distance between any two odd faces in* $T(G)$ *is* d *and the number of odd faces is* ω*. Can one always construct a cubic graph with these properties? 10. 96.*

Lemma. If G *is a planar cubic graph then* G *can be made an IOF by deleting at most* $n/4+1$ *many vertices.*

Proof: Let v *be a vertex belonging to more than one odd faces. Similarly as in the proof of the lemma,* ω *is lowered by 2 irregardless of whether* v *belongs to 2 or 3 faces.*

If a cubic oddette has n *vertices then* n *is divisible by 4.*

Indeed, if f_k *is the number of k-faces of an oddette then*

$3f_3 + 5f_5 + 7f_7 + ... = 3n$, *and*

$f_3 + f_5 + f_7 + ... = n/2 + 2$, *and thus*

$f_3 + 2f_5 + 3f_7 + = 5n/4 - 1$

Let δ *be the minimum number of vertices which have to be deleted to obtain a forest. Notice that we always have to delete at least* $f/2$ *vertices, because every reduction lowers the number of faces by at most 2. The equality holds true iff there is a sequence of deletions* $v_1..v_{f/2}$ *and graphs* $G_{k+1} = G_k \; v_k$ *such that* v_k *belongs to 3 faces of* G_k*. If* G *is cubic we to delete at least* $n/4+1$ *vertices.*

Let P *be the pentagon with 2 diagonals and let* G *be the graph obtained by attaching a copy of* P *to endpoints of the star with 3 edges. The graph shows that the number of needed deletions may be as large as* $3n/8$*. Is 3/8 the worst constant for cubic graphs? Is* $n/4+1$ *enough if* G *is 2-connected, or triangle-free? 10. 96.*

(*) In the case of fullerenes to get independence n/2 -c, with large c, one have to get examples of VIP - isomers with very large minimum distances between

pentagons. I think that such fullerenes can be constructed, but their existence still does not guarantee that c is small.

Such fullerenes indeed exist, see (**) below.

10. 96 What is the largest constant c s.t every n-vertex planar, triangle-free graph G has an induced bipartite graph with cn vertices. What if G is K_4-free? 10. 96.

For IOF cubic triangle-free graphs, I conjecture that they can be made bipartite by deletion of n/5 vertices, n/5, if correct, would be the best possible, because of IP isomer C_{60}.

Oddette is well defined, and is of interest for arbitrary graphs. Instead of odd faces the algorithm will work on elimination of minimum (i.e., chordless) odd cycles. The algorithm is formally similar to Maxine, so it seems natural to consider the following invariants and questions: let c(v) be the number of odd cycles, containing vertex v. Which sequences c can be realized in graphs? Perhaps this will lead to something similar to residue, comp 69.

An analog of Min is ***Evette*** *an algorithm which starts with the empty set B and the full set of undeleted vertices and then expands B by adding to it the first vertex v creating no odd cycles in B + v, and then delete all those vertices which do.*

I experimented in this version of Graffiti with an algorithm, which deletes first vertices producing maximum of horizontal edges, and DeLaVina used another algorithm related to 750 and 753, but hopefully Oddette and Evette will do better.

(*) Suppose that the minimum distance between pentagons is 2k. Let B_k be a ball around a pentagon P, i.e., the set of vertices whose distance from P is at most

k. It is easy to see that the number of vertices of B_k is of order 2^k, so the two closest pentagons can be separated by two balls of exponential order in k.

If the 12 pentagons can be paired so that distances between them are about k, the first lemma implies that G can be made bipartite by deletion of at most $c \times k$ vertices, and the number of vertices is at least $d\times\ 2^k$, where c and d are constants. This roughly shows that a fullerene can be made bipartite by deletion of clogn vertices. The argument is not complete, because after separating the first pair of pentagons, the minimum distance between the next two closest may be much larger. But in this case we can take even larger k, so even if the new separating balls overlap with the previous ones, the argument shows that a fullerene can be made bipartite by deletion of O(logn) vertices.

The argument gives only an upper bound for β. It is still possible that β and thus the independence number are of order n - constant. 4. 96.

(**) **Gunnar Brinkmann, University of Bielefeld, Charles Delorme, University of Paris- Sud** *and* **Tadeusz Januszkiewicz, Wroclaw University** *told me that one can have fullerenes with the shortest distance between pentagons being arbitrarily large. Brinkmann and Delorme refer essentially to Coxeter construction described in the "Atlas of Fullerenes", though Delorme's construction is more direct. Januszkiwicz deduced the answer from geometric considerations in [T].*

Spiral fullerenes are determined up by 12 integers - corresponding to 12 pentagons. I wonder if in general distances between vertices corresponding to pentagons determine a fullerene.

4. 98. I asked this question after Gunnar's talk at the Dimacs meeting last month, in which among other problems he discussed coding of fullerenes. Gunnar's *group has now programs which generate 400 fullerenes per second, so they should be able to find a counter-example or test this question. The simplest question of this kind is whether a fullerene is determined by the number of its vertices and the set of the distances between its pentagons. 4. 98.*

This would lead to interesting questions from the general point of view of Graffiti, where objects at certain level are represented by a finite (bounded, and preferably as short as possible) string of numbers, [ACC].) Some questions may be also of interest in complexity of computation: consider for example the following decision problem D.

Instance: 13 integers, 12 of which define a spiral fullerene G.

Question: Does G has an independent set $\geq$ thirteenth integer.

The question is whether D belongs to NP, and if yes, whether it belongs to P. The size of the problem is of course logk, where k is the largest of the 12 integers.

If fullerenes are determined by distances between their pentagons, then we would have a corresponding, (not necessarily more difficult) problem for arbitrary fullerenes.

Returning to relevance of consize representations in the program

[T] William Thurston, Shapes of Polyhedra.

844. *The independence number in fullerenes is greater or equal to the number of eigenvalues greater or equal to -1 (and it seems that there are quite a few graphs in which the equality holds true.)*

845. *The sum of positive eigenvalues of any fullerene G is not more than 1 + the number of vertices in a largest bipartite subgraph of G.*

According to some (see for example [CDS]) models, the energy of a molecule is expressed by the sum of its first n/2 eigenvalues. Since for bipartite graphs this *happens to be the sum of positive eigenvalues, I understand now better why Professor Matcha considered my conjecture described in 28 as plausible; it seems reasonable that the boiling point can be correlated with the energy. Since fullerenes are "almost bipartite", I wonder if a similar relation holds true for these graphs, and if there is data available to test it.*

The difference between the right and left sides of the inequality in 845 is larger for each of 9 unstable examples (see 840) than for each of the stable ones. I think that this remark and the conjecture itself adds some credibility to my speculation (*) from 840.

Similarly as for the independence number, I use an approximation algorithm for the largest bipartite subgraph, but I think that the rough pattern is correct. The similar (though not always so clear) pattern occurs sometimes in other conjectures and I will refer to it as a **stability-sorting** *pattern. In such conjectures the difference can be larger for stable examples, (in which case I will call the pattern* **negative***.) At the very least, the stability - sorting pattern will mean that for fullerenes with comparable number of vertices the difference between two sides of the inequality polarizes the 19 examples of Babic who selected them so that they more or less can be matched by the number of vertices. This function of Graffiti is not at the moment automated, because I never used the program in this capacity (though it is similar to heuristic Echo), so it is possible that I may overlook sometimes a counterexample, but overall pattern will be usually quite clear, whenever I plan to write about it.*

Echo selects as interesting those conjectures which are false in the background, i.e., a collection of graphs which do not have the property in question, so in this case, graphs which are not fullerenes. At the moment I use as a background less than 30 cubic graphs of girth 5. As a result of examining the stability-sorting pattern, I *noticed that some conjectures have much more pronounced let's call it* **characteristic pattern** *i.e they fail for many graphs in the background. I seem to remember that at an experimental stage one conjecture was false for every graph in the background, i.e it had a perfect characteristic pattern. A perfect pattern is like an iff conjecture. Actually I experimented with exactly this qualitative version of Echo in early versions of Graffiti, and now it is clear that I shouldn't have given up on it. The heuristic was measuring the probability with which the conjecture was false in the background. The difficulties with Echo were described in [G3]. 4. 96.*

[CDS] Spectra of Graphs,

[G3] On Conjectures of Graffiti, III,

846. *The sum of positive eigenvalues is $\geq$ 1 + the number of positive eigenvalues - smallest eigenvalue of the complement of a fullerene.*

comp. 20 and 856.

This conjecture has a negative stability-sorting pattern

847. *Let m be the minimum of responses of the second player (to all possible moves of the first one) in the market game described in 749. If G is a fullerene then m +2.7 $\leq$ the sum of positive eigenvalues.*

This conjecture has a negative stability-sorting pattern. For some reasons, conjectures about market game of fullerenes are more common than usually.

848. *The number of negative eigenvalues of a fullerene is not more than mean e(v), where e is the number of vertices at even distance from v.*

This and the next conjecture have a clear stability-sorting pattern as well as very strong characteristic patterns - they fail for more than half of graphs in the background.

849. *Let v be a vertex maximizing the number of horizontal edges. If G is a fullerene then the number of negative eigenvalues of G is not more than the number of vertices at even distance from v.*

This conjecture has the same patterns as 848.

850. *Let $w(v)$ be the number of vertices at odd distance from v, and let m be the minimum of w. If G is a cubic graph of girth 5 then the number of negative eigenvalues of G is not more than $1 + m$.*

851. *Let v be a boundary vertex of a graph, i.e, a vertex of maximum eccentricity, and let s be the number of vertices on a maximum sphere with the center at v. If G is a fullerene then its radius is $\geq s - 2$.*

This conjecture has very strong both characteristic and the stability sorting patterns.

852. *The number of negative eigenvalues of a fullerene $\geq$ the maximum of horizontal edges.*

853. *The average distance of a fullerene is not more than its radius -1.*

This conjecture has very clear negative stability-sorting pattern.

854. *The average distance of a fullerene is not more than the number of vertices in a mean interval. Intervals are defined in 772.*

855. *The number of positive eigenvalues of a fullerene is $\geq$ 2(maximum of horizontal edges - minimum of horizontal edges).*

This conjecture has a clear negative stability-sorting pattern.

856. *The sum of positive eigenvalues is $\geq 1 +$ the number of eigenvalues greater or equal to -1.*

comp 846 and 20.

857. *Let m be the number of (linearly independent) eigenvectors of 0 over the 2-element field. Eigenvectors of 0 of a cubic graph can be thought of as sets E of vertices such that every vertex has 0 or 2 neighbors in E.*

Let t be the length of the largest side of a largest triangle of G.

Conjecture: $m \leq t$.

This conjecture has exceptionally strong stability-sorting pattern. After sorting graphs by the difference between both sides of the inequality, the two largest stable examples appear on the beginning, and the two largest unstable at the end of the list.

858. *A vertex is a* **center** *if it has minimum eccentricity. Let $h_o(v)$ be the number of edges whose endpoints are at the odd distance from v, and h - the minimum of h_o.*

Conjecture: The number of centers of a fullerene is not more than n-3 + h.

I was surprised that Graffiti made such a weak conjecture, but apparently h can be quite small for fullerenes. Looking up the data I noticed that h itself, has a very strong negative stability-sorting pattern, which is the main reason for listing 858 here.

For all unstable examples h is between 1 and 3, and for all stable ones - between 6 and 9. The pattern seem to hold true also for minimum of horizontal edges and minimum of horizontal edges at even distance from v.

859. *If G is a fullerene then the minimum of horizontal edges is not more than $2 \times$ radius.*

This conjecture has a clear stability sorting pattern, with one notorious stable example again sticking somewhat out of the crowd.

May 96

860. *The sum of positive eigenvalues of an IP isomer G is not more than -1 + the number of vertices in a largest bipartite subgraph of G.*

comp 845. The difference is smallest in C_{60} *in the sample of about 70 graphs.*

861. *The sum of positive eigenvalues of an IP isomer is at least* $3n/4 + 1.6$.

862. *The independence number of an IP isomer is at least 1 + max (e(v) - h(v)), where e(v) is the number of vertices at even distance from v, and h(v) - the number of horizontal edges at even distance from v.*

750 (*) implies that apart from the summand 1, the conjecture is correct, or in other words 862 asserts that we can't have equality for fullerenes in 750 (*).

Let E(v) be the graph induced by vertices at even distance from v, where v is an optimal vertex. If G is cubic then E(v) has maximum degree 2. Thus a necessary condition for equality in 750 (*) is that every component has one vertex or one edge. This remark can be remade into a considerable improvement of 750 (*) in the case of cubic graphs. In general let c_k *be the covering number of the kth component of E(v), and let* $\eta(v)$ *= e(v) -* Σ c_k. *Then the independence number of G is* $\geq$ *max* $\eta(v)$. *In general this bound is of course difficult to compute, but in the case of cubic graphs - the problem - including the algorithm - is trivial, and it it seems that at least for fullerenes the algorithm would perform very well.*

Let δ *be corresponding bound with respect to vertices at odd distance. It would be interesting to know how large can be the difference between the independence number and maximum of* η *and* δ *for cubic graphs, cubic triangle-free graphs, girth 5, planar, etc. Of separate interest is performance of this bound for the corresponding random cubic graphs and random graphs of maximum degree 3.*

Conjectures about Ramseyan Properties

In conjectures below G is a triangle-free graph and red and blue graphs are defined with respect to this property (comp 822).

863. *Let G be a cubic connected graph of girth 5 with 16 vertices. Let us color red the pairs of vertices of distance 2 and blue otherwise, comp 822 -839. If the red graph of G contains no 4-element clique then the blue clique number is $\geq$ the red independent domination number.*

Let T(r,b) be the smallest integer such that every triangle-free graph with T vertices contains an r-element red clique or a b-element blue clique. **DeLaVina** *proved that T(4,b) = 8(b-1) +1, which seems to confirm the conjecture of DeLaVina, Erdos and myself in the (4,b)-case, see 822, but the description of critical graphs, needed to prove this conjecture, was far from the obvious even in the case of 16-vertex graphs. 863 was an attempt to generate a conjecture which would help to prove that there are no connected 16-vertex critical graphs T(4,3) and this indeed happened essentially the same day, in spite of the fact that 863 itself is still open. It is difficult to tell how helpful was 863, but I was working on critical T(4,3) graphs for about a week before and I knew some of the lemmas below beforehand. 863 may be still of interest because it solves the problem of critical T(4,3) graphs in view of Lemma 2. One of course could verify the 16 vertex case or 863 with a computer, but such a proof would be useless. Of course if one could just clearly delineate the difference between a straightforward verification and a proof (involving a finite problem) than this in itself could be a major accomplishment. A sensible solution would be to define a new kind of proofs, (like for example constructive proofs.), but in this case one would like to exclude a straightforward verification.*

I would consider such definition of a new concept significant, because most of the time, it is difficult to imagine a proof of results like say Brooks Theorem which *would work for all at most 100 vertex graph, but it did not work for all and which would not be a straightforward verification. Criteria for these special proofs might for example forbid a direct reference to the size of the object.*

Since DeLaVina already proved before that a connected T(4,3)-graph must have girth 5, I tried to generate conjectures about this case.

I will use the following notation and preliminary observations: Let v be vertex of G, A the set of its neighbors, B - the set of vertices at distance 2 from v, and let C be the set of remaining vertices. Note that B and C have 6 elements each, and every nonedge {v,c}, where c is an element of C, is blue. In particular we have that the blue graph of G is regular of degree 6 and

Lemma 1: *every vertex of C has exactly five neighbors in A+B.*

Lemma 2: *The red independent domination number is at least 3.*

Proof: If a and b are red independent then they are either adjacent in G, or their distance in G is at least 3. G has no two-element red independent dominating set consisting of two adjacent vertices since such vertices can dominate at most 14 others. Thus by contradiction we can assume that G has a red dominating set of the the form {v,c}, where c ϵ C.

Since neighbors of v are not adjacent to v in the red graph, if the red independent domination number were 2, they would have to be adjacent to c in the red graph. But since the neighbors of v are adjacent in the red graph, together with c they form a 4-element red clique.

Lemma 3 (DeLaVina.) *The graph induced by C has no isolated vertices.*

Proof: A vertex isolated in C would have 3 neighbors in B, and together with v they would form a 4-element red clique.

Lemma 4. *Let c be an element of C with two neighbors in B. Let b be a neighbor of c in C. Then all neighbors of b are in C.*

Proof. If c had two neighbors in B then they and their neighbors in A would be at distance at most two from c. Since G has no C_4 that would produce 4 vertices of $A + B$ at distance at most 2 from c.

Thus by Lemma 1, all remaining vertices of $A + B$, would have to be at distance 3 from v. Since G has no triangles, no neighbor b can not be in B.

Theorem. *Let G a graph satisfying assumptions of 863. Then G is a disjoint union of two copies of regular Ramsey $R(3,4)$ graph.*

Proof: If every vertex of C had at most one neighbor in B, then the graph induced by C would have at least 6 edges and thus every vertex of this graph would have degree at least 2. Thus the graph would have to contain a cycle, and since the girth is 5, it would have to be isomorphic to a C_6. Note that in this case every vertex of C is adjacent to exactly one vertex of B.

Since the distance (in G) between any two elements of C is at most two, every two antipodal vertices of C must be adjacent to the same vertex in B. Thus the degree sequence of the graph induced by B, is (0, 0, 0, 2, 2, 2) and thus this graph would contain a triangle which is contradiction.

Thus we can assume that one vertex of C has 2 neighbors in B, and in this case, by Lemma 4, it is adjacent to a vertex b in C whose all neighbors are in C. Since C contains no isolated vertices and girth 5, this leaves two possibilities: either the graph induced by C is a tree with the degree sequence (2, 2, 1, 1, 1, 1) or an amalgamat of pentagon with an edge.

The first case is exactly the same as of C_6, because the graph induced by C contains three pairs of vertices at distance 3. In the second case there are two pairs *of vertices of distance 3, and they show how two construct a* **unique** *graph extending the graph induced by C to a cubic graph of girth 5 satisfying constraints of the proof. But this graph contains a blue triangle. s.f. 5. 96. comp. 864 and 865.*

I guess I was right that one should not use a computer for straightforward verification of critical case of T(4,3) because it is clear that lemmas can be generalize to provide some general bounds for T(r,3). These numbers lead naturally to the following two problems.

G: *What is the smallest number n=g(d,k,l) such that every n-vertex graph of maximum degree d contains k vertices with mutual distances greater or equal to l?*

T: *Let G be a n-vertex graph (of maximum degree d) containing no k+1 - element set of vertices whose mutual distances are at least l. What is the maximum number of pairs of distance* $\geq l$*? Instead of pairs one can also ask about the number of p-element systems,* $p \leq k$*, of vertices with mutual distances at least l.*

These problems are clearly a motley of Ramsey, Turan and Moore problems.

Another example of Moore-like question which arose naturally in study of connected version of numbers T(r,b) is

M: *What is the maximum number of vertices of a triangle-free, diameter 2 graph of maximum degree d, which is not regular.*

The irregularity condition is motivated by the following example: Let S be the triangle-free, 7 vertex cycle with 3 diagonals. Taking two copies of S and joining by an edge two vertices of degree 2, we get an example of a connected T(4,3) graph. S appears later several times in conjectures involving blue clique, see for example 881.

To be of use in construction of T-critical these graphs must have also the property

M1: *the independence number is equal to d,* *which is of interest for its own sake, since one would like to have a characterization of graphs in which the red clique number is equal to the maximum degree, comp 867.*

An obvious bound for graphs satisfying M is 1 + d(d-1) and for d = 3 S is an example of equality. For d = 4 the bound however is 12, because of the uniqueness of Ramsey R(5,3) graph, which is regular. But even without the condition M1 for the independence number in order to achieve the 1 +d(d-1) bound, d must be even. Indeed, let us call the graph induced by vertices of distance 2 from the vertex of degree less than d the **inner graph of G**. *The inner graph must be regular of degree d-1, but the number of its vertices is* $(d-1)^2$, *so d must be odd.*

By Brooks theorem the chromatic number of the inner graph is at most d-1, but to qualify as an inner graph a regular triangle-free graph must have a (d-1) coloration with the property that

M2: *every pair of vertices of distance* $\geq$ *3 is monochromatic (with respect to a color)*

I think that colorations (and the resulting concept of chromatic number) satisfying M2 are of interest for their own sake since they are relaxation of the property of having diameter 2.

To satisfy the condition M1, a candidate inner graph must have the property

M3: *the independence number is equal to d-1.*

An example of a graph satisfying all these conditions is C_4, *which is the inner graph of the graph S above.*

Examples of triangle-free graphs satisfying M3 are discussed in 790. The existence of d^2*-element graphs of diameter two and maximum degree d was settled in [EFH] and partial results concerning the case* d^2-1 *are in [F] and perhaps one* *can use methods from these papers to decide existence of graphs satisfying M. It is not clear if one can use directly the matrix equation for adjacency matrix of Moore graphs since the matrix which plays the role of J, [HS], does not commute, with the adjacency matrix of the graph in question, or at least it is not obvious whether it does.*

Let us call a pair (G,γ) an **inner graph** *if γ is a coloration of G satisfying M2 and*

M4: *for every vertex v and color c not equal to $\gamma(v)$ there is a vertex u of color c adjacent to v.*

If (G,γ) is an inner graph then G is graph obtained from G by adding colors as vertices joined to vertices of their color and an "infinity" vertex γ joined to all of the colors. Graphs containing a vertex contained in no C_3 no C_4 induce inner graphs and often, as it is the case with Moore or M-graphs, they can be studied by means of "abstract" inner graphs. All possible γ colorations will often, or at least in many interesting cases, induce a a natural geometry structure on G by taking as lines vertices of the same γ color; it is known for example that Hoffman-Singleton graph arises from a projective geometry.*

For d = 4 there is 16 vertex example which can be obtained from a product of C_4 with itself, two pairs being adjacent if their first components are adjacent and second equal. the inner coloration can be obtained by matching independent sets in different components so that the resulting distances are equal to 3.

DeLaVina *found an example with 11 vertices which has 2 vertices of degree and which can be extended to a critical T(5,3) graph similarly as S, so of interest are also graphs which have more than one vertex of degree less than maximal. 6. 96.*

[EFH], Erdos, Fajtlowicz and Hoffman, Networks,

[F], Fajtlowicz, Colloquium Mathematicum.

[HS] Hoffman and Singleton.

* * *

864. *Let G be triangle-free graph. Then the red counter-independence number is $\leq$ the blue clique number.*

Any maximal blue clique is a red counter-independent set.

This conjecture has a similar potential as 863. **Ermelinda DeLaVina**. *5. 96.*

865. *Let G be a triangle-free graph. Then the blue clique number $\geq$ 1/2(red independence number.)*

The conjecture is correct because the red independent set consists of isolated vertices or edges whose endpoints are at distance at least 3. This implies that if the red independent domination is $\geq$ 5 then the blue clique number is at least 3. The same is true if the number is 4 unless the only red dominating set consist of two edges. In this case in 863 there are 8 vertices at distance 1 from these edges and if D is a minimum red independent dominating set then for each element d of D, among the remaining 4 vertices there is a unique vertex d of distance 2 from d. The distance of d* from remaining elements of D i s at least 3. Thus in this case there is a blue triangle and to prove that there are no 16-vertex connected T(4,3) graph it is enough to show this for graphs in which the red domination independence is 3.*

866. *The number of red components is $\leq$ twice the number of components.*

The conjecture is correct and equality holds true iff every component is bipartite.

867. *The red clique number is at least as large as the maximum degree.*

This conjecture is obvious but one would like to describe the graphs satisfying equality. The case of equality of upper bounds in terms of degree was settled by **DeLaVina** *in 838, see also 863.*

* * *

868. *The independence number of the blue graph is* $\geq$ *the minimum span of two nonadjacent vertices of G.*

869. *The independence number of the blue graph is* $\geq$ *sum of temperatures of vertices of G.*

870. *The jet number of complement of the red graph is* $\leq \pi(n)$ *-the number of primes less or equal to the number of vertices.*

I think that the worst case for large n should be Ramsey graphs R(k,3) for large k.

871. *If G is a cubic connected triangle-free graph an m is the smallest integer* $\geq n/4$ *then the blue clique number is is* $\leq$ *m - the size of the smallest independent dominating set of G.*

June 96.

Conjectures below are an attempt to generate properties or lemmas which may be useful in proving conjectures, similarly as it was the goal of 863. I do not mean necessarily proving conjectures by humans. Bledsoe was, I think first, to point out that a progress in automated theorem proving may require automated conjectures. Some conjectures below may be useful in splitting problems into cases, and then attempting to find new conditional conjectures, until they are obvious.

Conjectures below are by searching for bingo conjectures, [ACC] whose goal is an acceleration of computational processes. The first few conjectures are actually bingos, though here an attempt to accelerate the process is meaningless, since the *invariant in question - the slow variable in terms of [ACC] is the degree d of a regular graph. Instead of speed-up the issue here is information content provided by a conjecture, and splitting the problem into cases.*

* * *

872. *Let G be a d-regular, connected triangle-free graph. Let us delete a vertex of minimum degree, note its degree g_i and continue the process. Let m be the mean of the resulting sequence, and let us call it the mean global minimum. Then m equals to d/2.*

At first, this conjecture looked unbelievable to me, undoubtedly because it has unnecessary assumptions, comp. conjecture -1. The conjecture is true because deleting a vertex of degree g_i we remove g_i edges and the number of edges is nd/2. Actually the stronger conjecture that average g_i is half of the average degree is even more trivial, but nevertheless this conjecture has obviously an information content, whatever the term means. 872 is very close to the statement that the number of edges is half of the sum of degrees, and this statement is used all the time, as well as its corollary about parity of vertices of odd degree.

* * *

873. *Let r be the counter-independence number of the complement of red graph. If G is a cubic connected triangle-free graph of girth 5 then either diameter is 2 or r is 2.*

Graffiti's formulation was that 1 + r = the degree of the graph.

We have an obvious

Lemma: Let G be a d-regular graph of girth $\geq$ 5. A singelton v is counterindependent in the complement of red graph iff S(v,2) - set of vertices at distance 2 from v forms a red clique, i.e. every two vertices in this set have distance 2 (in G.)

A **sphere** *$S(v,r)$ is the set of all vertices whose distance from v is r, $r \geq 0$.*

Suppose that the red counterindependence number of the complement of the red graph is 1. Since girth ≥ 5, and G is cubic, for every v, $S(v,2)$ has six vertices. Three pairs of these vertices have (already) distance 2 in G, because they are adjacent to the same vertex adjacent to v. For each of remaining 12 pairs (x,y) there is a vertex $s = s(x,y)$ of distance 3 from v, adjacent to both x and y. Since every s can handle at most three pairs, G must have 4 vertices of distance 3 from v, and its diameter must be 3. It is easy to see that there is unique such graph, and it must be isomorphic to Heawood graph, whose girth is 6. comp. 877.

This proves the inequality in 873, but I do not know if 873 is correct.

Suppose now G is a regular graph of girth 6 in which counterindependence number is 1. Let P and L be as in 838. From the theorem of **DeLaVina** *in 838 it follows that (P, L) is a projective geometry.*

It is easy to construct graphs of girth 4, in which the counterindependence of the complement of the red graph is 1. The simplest examples are $K(3,3)$ and a cube.

* * *

874. *A sphere $S(v,r)$ is the set of all vertices whose distance from v is r, $r \geq 0$. A sphere is* **central** *if v is a center of G.*

Conjecture:

If G is a cubic, connected, triangle - free graph, then one of the following three conditions is true:

A: half of the maximum span of two nonadjacent vertices is d.

B: d - the degree of the graph = 1 + the number of components of the red graph.

C: the second smallest central sphere has at least d vertices.

The conjecture was made first by Graffiti under an additional condition that d equals to the red clique (this version does not use Echo,) but I did not settle it until I proved 875.

*Proof. Suppose that G does not satisfies A. Then G has diameter 2. If G does not satisfies C then by 875(*), G is complete bipartite, (note that the C involves a tryout, but it does not matter since 875*C is stronger) and then by 866, G satisfies B.*

875. *Let G be a cubic, connected, triangle-free graph. Then one of the following three conditions is true:*

A: half of the maximum span of two nonadjacent vertices is d.

B: d = 1 +the independence domination number of complement of the red graph.

C: the second smallest central sphere has at least d vertices.

The conjecture is correct. I'll prove a slightly stronger statement:

(*) If G is a d-regular, connected, triangle-free graph then either:

(a): G has diameter $\geq$ 3,

(b): G is isomorphic to $K_{d,d}$, or

(c): for every vertex v and r $\geq$ 1, S(v,r) has at least d vertices.

(*) is stronger than the original conjecture, because for every regular graph A is equivalent to (a), and the two remaining lower case conditions are stronger, i.e., more restrictive than the corresponding upper case conditions.

Proof: Suppose that G is a graph of diameter 2, which does not satisfy c. Then for some vertex v, the number of vertices of $S(v,r)$ is $\leq$ d-1. If G does not satisfy a, then r must be 2. Since $S(v,1)$ is independent the number of edges between these two sets is d(d-1). Thus S_2 must have d-1 vertices and they also must be independent, *by the edge count. Since v is not adjacent to any vertices of* $S(v, 2)$*, G is bipartite, and since it has 2d vertices, and* d^2 *edges it is complete bipartite.*

* * *

876. *Let G be d-regular, triangle-free connected graph and let* $\omega(v)$ *be the number of vertices at odd distance from v - the number of horizontal v-edges at odd distance from v. Then* $d \leq$ *minimum of* $\omega(v)$.

This conjecture is almost certainly false for large d, but it is true (and bingo hunt suggests that it may be useful) for cubic graphs. I think that the conjecture should be false for d = 4.

Proof: Let d=3, and let S_r *= S(v,r) be as in 874. The graph induced by* S_r *has maximum degree 2, and thus it can not have more edges than vertices. Since* S_1 *contributes d to* $\omega(v)$*, the conjecture follows. In the case of equality the graph induced by* S_3 *must be a union of cycles, and thus diameter of G is either 2 or 3* S_4 *is empty because every vertex in* S_3 *already has degree 3.)*

877. *Let G be a cubic, triangle-free graph, s the number of vertices in the minimum span of a pair of nonadjacent vertices and r - the counterindependence number of the complement of the red graph.*

In several attempts to generate a bingo conjecture, Graffiti listed as a one of the conditions that

(*) s - r $\leq$ *3,*

and it revealed that there are many graphs with equality. I skip these assumptions in view of

Theorem: s - r $\leq$ *3, with exception of Heawood graph,*

which proves most of these conditional conjectures.

Proof: $s \leq 5$, and s=5 iff girth is ≥ 5. Thus if s -r ≥ 4 then girth is 5 and r = 1, (r is 0 iff G has no edges.) For the same reasons as in 873, G is the Heawood graph (and for larger d's a similar argument should imply that G is the projective geometry graph.)

878. *If G is a cubic graph triangle-free graph then the red clique number is $\geq$ 1 + counter-independence of complement of the red graph.*

The argument from 873 shows that if the red clique number is 7 then r - the counterindependence of the red graph is 1. In general let s be the number of vertices of S(v,2) and let R(v) be the number of vertices of the largest red clique C containing v. Then

(*) $r \leq s - R(v)$.

Indeed, for every vertex x in S(v,2), outside of C, there is a vertex c(x) in C, such that the distance from x to c(x) is different from 2. The set of all c(x) is a counterindependent set of complement of the red graph. This proves (*), and for cubic graphs this implies that a counterexample to 878 must have the red clique number equal ≤ 4. Petersen graph is the only example of a cubic graph, I know of, in which r is ≥ 3.

* * *

879. *If G is a cubic, triangle-free graph then the independence number is not more than the number of its red components + mean of global minimum of the complement of G (the last invariant is defined in 872.).*

By 866 and 872, this conjecture is equivalent to the statement that the independence is at most n/2, and unless G is bipartite the independence is at most n/2 -1. This is obviously true for all regular graphs of odd degree.

It would be interesting to have a simple characterization of regular graphs in which independence is equal to n/2 -1. This is obviously related to 750 and many related conjectures including some about fullerenes, see for example 840. Note that the maximum number of odd disjoint cycles in such graphs is 2. Good algorithmic characterization is easy, because if independence is n/2 -c then there is c1, such that G is bipartite after removal of c1 many edges.

* * *

880. *A* **connector** *is a vertex which is not a cut-vertex. If G is cubic triangle-free graph then its independence number is $\leq$ than half of the number of connectors.*

880 is false. It isn't too hard to construct a cubic triangle-free graph G with 28 vertices of which 24 are connectors but where G has 13 independent vertices. But 880 leads me to the interesting observation that cubic bipartite graphs can't have cut vertices. Them same should be true about bipartite graphs with odd degrees of regularity. **William Staton.** *6.96.*

comp 751.

The same can be done with $n = 22$, which produces the independence/connector ratio 5/9, which perhaps is the best possible. In general it seems of interest what is the smallest constant for graphs of a given maximum degree and girth.

Concerning the observation about bipartite graphs, it led me immediately to the question, what is the best upper bound for the independence number which can be obtained from the degree sequence of a graph. Perhaps it can be done in a manner similar to residue, see 69, using instead of maximum, the minimum degree d, according to Graffiti's suggestion that the independence number is not more than n - d, conj 158. It is of course not clear at all how to define the step corresponding to the Havel-Hakimi reduction. s.f. 6.96.

881. *If G is a cubic triangle-free graph then its independence number is $\geq$ than its diameter.*

881 is false. Take a 6-cycle with two polar diagonals. String together a bunch of these and then finish the two ends with 7-cycles with 3 diagonals. It seems to me that this example is also relevant to 880. the ratio of independence number to connectors is as close as you like to 3/4. **William Staton.** *6.96.*

(*) The ratio of independence number to diameter can be made as close to 1/2 as you like with any maximum degree you like. The question is perhaps more interesting if one excludes triangles. In that case I think I know how to increase the ratio a bit. For cubic triangle-free I suspect the ratio is asymptotically 3/4. **William Staton.** *6.96.*

Let G have diameter d and minimum degree k. If G is triangle-free then the independence number is at least (kd+2)/4. We have examples to show this is asymptotically best possible. In case the girth is bigger than 5 we can do better. **Glenn Hopkins, and William Staton, OleMiss.** *6.96.*

The theorem for triangle-free graphs is a little better than I thought. Independence number $\geq$ (kd+k+2)/4 where k is the minimum degree and d is the diameter. I want to know how much better this could be if we assume there is no cut vertex. **W. Staton** *6.96.*

If X is a set of vertices of G then $B(X,k)$ is the set of vertices of distance k or less from an element of X. Let d be an integer and c a family of paths such that for every P, and Q belonging to c we have:

1. *if v is in B(P,2) and B(Q,2) then v doesn't belong to B(P,1) nor B(Q,1).*
2. *If v belongs to B(P,1) then its degree in B(P,2) is greater or equal to d.*

3. The distance between endpoints of P in B(P,2) is p -1 , where p is the number of vertices of P.

We define now h(P) to be the sum of lengths of paths P in c, and the **d-area** *of G the maximum of h(c) over all possible c's satisfying 1 - 3.*

The definition is obviously motivated by an attempt to get most of the above results. s.f. 6. 96.

*If G is cubic and 3-connected then independence $\geq$ diameter If in addition G is triangle-free there is a constant c ¿ 1 such that independence   c*diameter.*

I believe that independence $\geq$ diameter should be true for cubic triangle-free 2-connected graphs but I don't have a proof. I also have some sharp results for larger degrees along with Glenn. **W. Staton** *7.96.*

882. *If G is a regular triangle-free graph then its independence number is $\geq$ the independence number of its red graph.*

A previous version made a conjecture that for cubic graphs the left side is $\geq$ 1 + right side. This is true for every d $\geq$ 3.

A set is red-independent in a triangle-free graph iff the distance between any of its two vertices is 1 or $\geq$ 3.

Since G is triangle-free, G-components of a red independent set are single vertices or edges. Let K be any set of edges of a red independent set, and let K be the set of vertices at distance 1 from K. The graph induced by K+ K* has maximum degree d, and 2kd vertices. Since it is triangle-free it follows from known results (for example from Staton's 5/14 Theorem) that the independence is strictly more than 2k. The maximum independent set of K + K* forms an independent set together with single vertices of the red independent set.*

The proof still leaves a possibility of equality if K is empty, but in the case of $d \geq 2$, one can replace a vertex in a maximum red independent set by its neighbors to get a larger independent set in G.

It may be of interest to have a direct proof of 882, particularly for $d = 3$.

In this case the graph induced by K^* is a union of cycles and paths , and unless some of the cycles are odd, the conjecture is obvious. Suppose that one of components C of K^* is an odd cycle. Let v be a vertex of C, and let $e = (x,y)$ be an edge of K, such that v is adjacent to x. Let K_1 be the set obtained from K^* by deleting the neighbors of v and y. K_1 has k-4 vertices, and the theorem follows by induction on the number of odd cycles (note that we did not assume that K is the set of all edges of the red-independent set.)

The above proof may be of interest in algorithmic search for independent sets. One can first find a large red independent set and then remake into an independent set of G. Since the former is smaller, as a rule, it will be easier to find. The best strategy may be simply to find a maximum independent set in which k is maximum. I think that usually the maximum independent sets should have a few singletons. 6. 96.

The independence number of G is clearly greater than the clique number of the red graph. Apart from Ramseyan numbers, each property defines also another generalization. For each property and color, say red, we can ask what is the smallest $n(k)$ such that for every m-vertex graph with this property, and m $n(k)$, the red graph of G has a red k-element clique or a red k-element independent set. This conjecture provides a bound for this number. 882 is not true, without the assumption of regularity, but perhaps the critical graphs must be close to regular as ordinary Ramsey critical graphs. 6. 96.

Of separate interest seem the smallest numbers (k,m) such that every connected graph with that many vertices has m-element set whose every two vertices have distance at most k, or a m element set whose every two vertices are at least k+1 apart. For $k = 2$, this the number with respect to triangle-free graphs and the blue color. These numbers are also related to Moore bounds, comp. 892. One can also consider the obvious 4-parameter version of this problem. Like for classical Ramsey numbers, I would expect that that best simple lower bounds should be provided by random graphs with the right probability p.

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883. *Let G be a regular triangle-free graph of diameter d. Then the red independence number is at least (1+d)/2.*

Proof: An induced path of length d has large enough red independent set. The worst case occurs when $d = 3 \bmod 4$, and in this case we can have equality. comp (*) in 881.

884. *Let G be a regular triangle-free graph of degree d. Then the red independence number is at most n/d.*

This is true because a r-vertex red independent set spans at least rd vertices in a regular triangle-free graph. The corresponding lower bound will be usually n/d(d-1), because d(d-1) is the maximum degree of the red graph, so the interval of values of red independence number has length about n/d.

If equality occurs then every maximum red independent set is a matching, which brings up a question whether maximum matchings in which distances between edges are at least 3 can be found in a polynomial time, or whether, and this seems to me more likely even in case of red graphs of regular triangle-free graphs, the problem is NP-hard.

885. *The red independence number is at most twice the maximum blue clique.*

This is obvious and if equality holds then as in 884, every maximum red independent set is a matching.

The program made false conjecture

885a. *that the red independence number is at least as large as the blue clique number.*

and

885b. *If G is regular of degree d then the blue clique number is at most n/(d+1).*

DeLaVina *noticed that we have a corresponding lower bound* $n/(1+d^2)$*. 6. 96.*

Her proof was more direct but here is an argument which is of interest, because it relates the problem to Moore and almost Moore graphs:

The maximum degree of the complement of the blue graph is not more than $1+d^2$ *which and thus the inequality follows from the trivial part of of the Brooks' Theorem. If equality holds the the complement of the blue graph must be complete, i.e., G has has diameter 2, and in particular the equality holds true iff, G is a Moore graph.*

Bill Staton and Glenn Hopkins make use of "Moore property" of Petersen graph in 887 and DeLaVina describes there a better bound. 7. 96.

* * *

886. *If G is a regular triangle-free graph then the red independence number is* $\geq$ *radius or G is bipartite and in this case the red independence is at least half of the radius.*

This conjecture made in the form that the red independence is $\geq$ *radius/number of components of the red graph. The conjecture is is similar to 883, because graphs of radius r contain an induced path with 2r -1 vertices.*

887. *Let G be a cubic (connected) triangle-free graph and m the smallest integer greater or equal to n/4 -2. Then G has a blue clique with m vertices, i,e a m-vertex set with every two vertices at distance 3 or more.*

This conjecture is very unlikely to be true in view of the upper bound from 885b.

Take copies of the Petersen Graph and remove an edge from each. Then string these guys together into a" cycle " . The result has no more than n/5 vertices mutually separated by distance at least 3. **Glenn Hopkins, and William Staton, OleMiss.** *7.96.*

The idea can be used for higher degrees, starting with any graph of diameter 2, and the closer the number of vertices to the Moore bound the better, comp 885. s.f. 7. 96.

DeLaVina *independently also found counterexamples to this conjecture, the smallest of which has 30 vertices and independence 5. She also proved that for d-regular graphs and large n, the blue clique number is at least* $n/(d^2-1)$ *- small constant and she found arbitrarily large cubic graphs in which the blue clique is at most n/6.*

888. *If G is a regular (connected) triangle-free graph and a the average distance of G, then G has a blue clique with m-1 vertices.*

July 96.

889. *If G is a regular (connected) triangle-free graph, w(v) the number of vertices at odd distance from v, and w - the maximum of w(v) then then G has a blue clique with w/4 vertices.*

The database contains just one example with equality and it is a 14 vertex graph described below condition **M** *in 863.*

890. *If G is a regular (connected) triangle-free graph, and r the residue of the complement of the blue graph then the blue clique number is at least r.*

There are several examples in which both invariants are equal.

891. *If G is a regular triangle-free graph then the blue clique number is greater or equal to the number of elements of a minimum spanning set minus residue, (the residue of a cubic graph is the smallest integer greater or equal to $n/4$.)*

* * *

891a. *If G is a regular triangle-free graph then the blue clique number is greater or equal to the residue of the complement of the blue graph.*

This conjecture is a special case of 69, but it is listed here because there are quite a few cubic graphs for which we have equality, and perhaps the case of equality can be characterized. By comparison the case of equality in 69 seems hopeless, and in the case of cubic graphs it is trivial.

Since the maximum degree of the complement of the blue graph is 9 this conjecture may be an improvement of DeLaVina's comment in 885b, but it is not obvious to me that this is case.

* * *

892. *If G is a cubic triangle-free graph then the blue clique number is greater or equal to the number of components of the graph induced by a minimum spanning set.*

The number of components of minimum spanning set is a try-out, and in this case perhaps even the weakest interpretation is of some interest.

* * *

893. *Let G be a regular triangle-free graph and s the number vertices in a minimum spanning set. Then the blue clique number is greater or equal to -1 +s/2.*

894. *If G is a cubic triangle-free graph then the blue independence number is smaller or equal to twice its girth.*

A set S of vertices is blue independent if the distance between any two elements of S is at most 2. At first one can think that the blue independence in a graph of maximum degree d is at most $1 + d^2$*, but it is not obvious that it is so because two vertices in S can be joined to the same vertex outside of S.*

Nevertheless this and other conjectures seem to indicate that the blue independence will be usually small (with respect to a given degree), which is of interest, because one can use the bound from [F]

$$a/n \geq 2/(p+q)$$

,

to get a lower bound for the size of a largest blue clique. Here a is the independence number of a graph, p - its maximum degree and q - the largest integer such that G contains no K_q.

For example it is fairly easy to verify (using Ramsey R(3,3) =6, but I should check it carefully) that the blue independence in fullerenes is 5. Let us use the bound (*) for the complement of the blue graph. Then p = 9, q = 5, and (*) implies that every fullerene has a set with n/7 vertices, every two being at least three apart. In the unique 24 vertex fullerene the blue clique is 4, so the best constant is between between 1/7 and 1/6.

Let S be a set vertices of a graph of maximum degree d, and suppose that diameter of S is 2, (in general the question is of interest for any diameter D and *the problem becomes another generalization of Moore bound.) The question is how large can be S.*

Let us join two vertices in S by an edge if they are adjacent in G, or there is a vertex in S adjacent to both of them, and let us call the resulting graph S(G).

If I is an independent subset of G(S) then there is a family L (partial projective geometry-like) of subsets of I (sets of neighbors of vertices outside of S) such that:

1. every element of L has at most d elements.

2. every element of I is contained in at most d elements of L.

3. every 2-element subset of I is contained in an element of L.

The third condition is the most interesting. To estimate the size of I we can use the above proof using the assumption that $\alpha_2 = 0$*. Assuming that girth* ≥ 5 *we have*

3a. every 2-element subset of I is contained in a unique element of L.

[F] Siemion Fajtlowicz, Independence, Clique Size and Maximum Degree, Combinatorica, Vol 4, Number 1, 1984.

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