A Proof of Graffiti 292

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Abstract

Let G be a connected simple graph of order n and girth at least five. Its gravity matrix is

$$\mathrm{Gr}_{uv} = \begin{cases} 0, & u = v, \\ \dfrac{d(u)d(v)}{(n-1)d(u,v)}, & u \neq v, \end{cases}$$

and its mean is taken over all n^2 entries. Graffiti 292 states that the least positive adjacency eigenvalue of G is at most $n/\operatorname{mean}(\mathrm{Gr})$. We prove the conjecture by combining a degree-sum upper bound for mean gravity, the trace bound $\lambda^2 \leq 2m$, and the standard extremal edge bound for graphs of girth at least five. Small orders are verified exactly.

1 Statement

For a graph G, let $\lambda_+(G)$ denote its least positive adjacency eigenvalue. Graffiti 292 is

$$\lambda_+(G) \leq \frac{n}{\overline{\mathrm{Gr}}}$$

whenever G has girth at least five. The statement is read directly from page 80 of *Written on the Wall*; the gravity definition is the one reproduced by Roucairol and Cazenave from Brewster–Dinneen–Faber.

Lemma 1. *If G has n vertices and m edges, then*

$$\frac{n}{\overline{\mathrm{Gr}}} \geq \frac{(n-1)n^3}{4m^2}.$$

Proof. Since $d(u,v) \geq 1$ for $u \neq v$,

$$\begin{aligned} \overline{\mathrm{Gr}} &= \frac{1}{n^2(n-1)} \sum_{u \neq v} \frac{d(u)d(v)}{d(u,v)} \\ &\leq \frac{1}{n^2(n-1)} \sum_{u \neq v} d(u)d(v) \\ &\leq \frac{1}{n^2(n-1)} \left(\sum_u d(u) \right)^2 = \frac{4m^2}{n^2(n-1)}. \end{aligned}$$

Taking reciprocals gives the result. □

Lemma 2. *Every adjacency eigenvalue λ satisfies $|\lambda| \leq \sqrt{2m}$.*

Proof. If $\lambda_1, \ldots, \lambda_n$ are the adjacency eigenvalues, then $\sum_i \lambda_i^2 = \operatorname{tr}(A^2) = 2m$. $\square$

Lemma 3 (standard C_4-free edge bound)**.** *If G contains no four-cycle, then*

$$m \le B(n) := \frac{n}{4}\left(1 + \sqrt{4n-3}\right).$$

Theorem 1. *Graffiti 292 is true.*

Proof. Suppose first that $n \ge 7$. Put $t = \sqrt{4n-3}$, so $n = (t^2+3)/4$ and $t \ge 5$. Because the left side below decreases with m and the right side increases with m, Lemma 3 reduces the desired comparison to

$$\frac{(n-1)n^3}{4B(n)^2} \ge \sqrt{2B(n)}. \tag{1}$$

Both sides are positive. Squaring (1), substituting $n = (t^2+3)/4$, and clearing positive denominators reduces (1) to $P(t) > 0$, where

$$P(t) = 2(t+1)^2(t^4 - 4t^3 - 2t^2 - 12t + 1).$$

Writing $t = u + 5$ gives

$$P(u+5) = 2u^6 + 56u^5 + 632u^4 + 3600u^3 + 10400u^2 + 12480u + 1152 > 0$$

for every $u \ge 0$. Hence, by Lemmas 1–3,

$$\frac{n}{\overline{\overline{\mathrm{Gr}}}} \ge \frac{(n-1)n^3}{4m^2} \ge \sqrt{2m} \ge \lambda_+(G).$$

For $n \le 6$, every connected simple graph of girth at least five is among the finite graph-atlas cases checked by exact characteristic polynomials and rational root isolation in the accompanying verifier. (The verifier checks all qualifying graphs through order seven.) $\square$

Reproduction

Run `python attack_graffiti_292.py`, `python verify_symbolic_292.py`, and `python verify_small_cases_e`
SHA-256 hashes are supplied in `SHA256SUMS.txt`.

References

[1] S. Fajtlowicz, *Written on the Wall*, July 2004, p. 80.

[2] M. Roucairol and T. Cazenave, *Refutation of Spectral Graph Theory Conjectures with Search Algorithms*, arXiv:2409.18626, 2024.

[3] S. Jukna, *Extremal Combinatorics*, section "Graphs with no 4-cycles."