Cyclic Rigidity in Dimension Two: A Newton-Edge Obstruction for a Pseudoreflection Normal Form

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Abstract

Let k be a field of characteristic zero, let $m \geq 1$, and let $H, g \in k[u, y]$. We prove that

$$muH_u g_y - H_y(g + mug_u) = c \in k^\times$$

forces $g \in k^\times$. Consequently every polynomial map

$$F(x, y) = \big(H(x^m, y), xg(x^m, y)\big)$$

with nonzero constant Jacobian is a polynomial automorphism and admits an explicit inverse. After swapping the target coordinates, maps in this class commute with the cyclic pseudoreflection $(x, y) \mapsto (\zeta x, y)$. Conversely, when $m \geq 2$ and after extending scalars to contain a primitive m-th root of unity, every polynomial map commuting with this action has the swapped normal form. The proof chooses the maximal Newton slope of g, isolates the corresponding weighted edges of g and H, and derives a one-variable edge polynomial whose terminal coefficient cannot vanish in characteristic zero. The result is unbounded in degree and allows arbitrary multimode support. It is a rigidity theorem for this normal form, not a resolution of the unrestricted two-dimensional Jacobian Conjecture.

1 Introduction

For polynomials $P, Q \in k[x, y]$, write

$$\mathrm{Jac}(P, Q) = P_x Q_y - P_y Q_x.$$

The two-dimensional Jacobian Conjecture asks whether $\mathrm{Jac}(P,Q) \in k^{\times}$ forces (P,Q) to be a polynomial automorphism when k has characteristic zero. We study the normal form

$$F(x,y) = \big(H(x^m,y), xg(x^m,y)\big), \qquad H,g \in k[u,y], \quad u = x^m. \tag{1}$$

There is a minor but important representation-theoretic distinction. Let

$$\rho_s(\zeta) = \mathrm{diag}(\zeta,1), \qquad \rho_t(\zeta) = \mathrm{diag}(1,\zeta), \qquad \zeta^m = 1.$$

Then the map in (1) satisfies

$$F \circ \rho_s(\zeta) = \rho_t(\zeta) \circ F.$$

It does not in general commute with one identical diagonal action. Equivalently, after the target-coordinate swap $S(P,Q) = (Q,P)$, the map

$$\widetilde{F} = S \circ F = \big(xg(x^m,y), H(x^m,y)\big) \tag{2}$$

commutes with $\mathrm{diag}(\zeta,1)$. The coordinate swap changes the Jacobian only by a sign.

Miyanishi proved an equivariant Jacobian theorem over $\mathbb{C}$ for small finite subgroups of $\mathrm{GL}(2,\mathbb{C})$ of even order [4]. The pure cyclic group generated by $\mathrm{diag}(\zeta,1)$ is generated by pseudoreflections and is not small, so that main theorem does not directly subsume the class (2). Our argument is instead an elementary polynomial Newton-edge argument over an arbitrary characteristic-zero field. Classical Newton-polygon, degree-reduction, and geometry-at-infinity results remain essential context [1, 7, 2, 5, 3, 6].

2 The cyclic Keller equation

Differentiating (1) gives

$$(F_1)_x = mx^{m-1}H_u, \qquad (F_1)_y = H_y,$$
$$(F_2)_x = g + mug_u, \qquad (F_2)_y = xg_y.$$

Hence

$$\det JF = muH_ug_y - H_y(g + mug_u). \tag{3}$$

Our main theorem is the following.

Theorem 2.1 (Cyclic rigidity) *Let k be a field of characteristic zero, let $m \geq 1$, and let $H, g \in k[u, y]$. If*

$$muH_u g_y - H_y(g + mug_u) = c \tag{4}$$

for some $c \in k^\times$, then $g \in k^\times$.

Corollary 2.2 (Classification and inverse) *Under the hypotheses of 2.1, there are $a \in k^\times$ and $h \in k[u]$ such that*

$$g = a, \qquad H(u, y) = -\frac{c}{a}y + h(u).$$

Thus

$$F(x, y) = \left(-\frac{c}{a}y + h(x^m), ax\right)$$

is a polynomial automorphism. For target coordinates $(P, Q) = F(x, y)$,

$$F^{-1}(P, Q) = \left(\frac{Q}{a}, -\frac{a}{c}\left[P - h\left(\left(\frac{Q}{a}\right)^m\right)\right]\right).$$

3 Boundary data

Lemma 3.1 *If (4) holds, then for some $a \in k^\times$ and $b \in k$,*

$$g(0, y) = a, \qquad H(0, y) = -\frac{c}{a}y + b.$$

Proof. Set $u = 0$ in (4). Then

$$-H_y(0, y)g(0, y) = c.$$

Both factors lie in $k[y]$ and their product is a nonzero scalar. Since the units of $k[y]$ are precisely $k^\times$, both factors are constant and nonzero. Integration in y gives the formula for $H(0, y)$. □

4 The maximal Newton slope

Assume for contradiction that g is nonconstant. By 3.1, every nonconstant term of g contains a positive power of u, so write

$$g = a + \sum_{j \geq 1} a_j(y)u^j, \qquad a \in k^\times.$$

Define

$$\rho = \max_{a_j \neq 0} \frac{\deg_y a_j}{j} = \frac{p}{q}, \tag{5}$$

where $p \geq 0$, $q \geq 1$, $\gcd(p, q) = 1$, and $0/q$ is reduced to $0/1$. Give $k[u, y]$ the rational weight

$$\mathrm{wt}(y) = 1, \qquad \mathrm{wt}(u) = -\rho = -\frac{p}{q}.$$

Every monomial of g has weight at most zero.

Lemma 4.1 (Edge of g) *Let $g_{[0]}$ denote the weight-zero component of g. Then*

$$g_{[0]} = A(t), \qquad t = u^q y^p,$$

where $A \in k[t]$, $A(0) = a \neq 0$, and $D := \deg A \geq 1$.

Proof. A monomial $u^j y^d$ has weight zero exactly when

$$qd = pj.$$

Coprimality gives $j = qn$ and $d = pn$ for some $n \geq 0$, so every weight-zero monomial is t^n. Because the maximum in (5) is attained by at least one $j \geq 1$, a positive power of t occurs. The case $p = 0$ gives $q = 1$ and $t = u$. □

Let σ be the largest weight occurring in H. This exists because H has finite support. By 3.1, the monomial $-(c/a)y$ occurs, so

$$\sigma \geq 1. \tag{6}$$

Lemma 4.2 (Edge of H) *There are integers r, s and a nonzero polynomial $\Phi \in k[t]$ such that*

$$0 \leq r < q, \qquad s \geq 1, \qquad H_{[\sigma]} = u^r y^s \Phi(t),$$

and

$$\sigma = s - \frac{p}{q} r. \tag{7}$$

Proof. If $u^j y^d$ and $u^{j'} y^{d'}$ both have weight σ, then

$$q(d - d') = p(j - j').$$

Thus $j - j'$ is divisible by q and $d - d'$ by p. All u-exponents on the edge have one common residue $r \in \{0, \ldots, q-1\}$ modulo q. For every edge exponent pair (j,d), write $j = r + qn$. Since $j \geq 0$ and $0 \leq r < q$, necessarily $n \geq 0$. Put $s = \sigma + (p/q)r$. The edge equation $d - (p/q)j = \sigma$ gives $d = s + pn$, so $s = d - pn \in \mathbb{Z}$ and

$$u^j y^d = u^r y^s (u^q y^p)^n.$$

Summing the finitely many edge monomials gives the displayed factorization with $\Phi \in k[t]$. Finally, $s = \sigma + (p/q)r \geq 1$. $\square$

5 The edge equation

Differentiation lowers weight by $\mathrm{wt}(u)$ or $\mathrm{wt}(y)$, while multiplication by u restores the loss from ∂_u. Therefore a pair of homogeneous components of weights α in H and β in g contributes to the Keller operator with weight $\alpha + \beta - 1$. Since $\alpha \leq \sigma$ and $\beta \leq 0$, the weight-$\sigma - 1$ component comes only from $H_{[\sigma]}$ and $g_{[0]}$.

Write

$$H_{[\sigma]} = u^r y^s \Phi(t), \qquad g_{[0]} = A(t), \qquad t = u^q y^p.$$

Then

$$(H_{[\sigma]})_u = u^{r-1} y^s \big(r\Phi + qt\Phi'\big),$$
$$(H_{[\sigma]})_y = u^r y^{s-1} \big(s\Phi + pt\Phi'\big),$$
$$(g_{[0]})_u = \frac{qt}{u} A', \qquad (g_{[0]})_y = \frac{pt}{y} A'.$$

Substitution into the Keller operator yields

$$mu(H_{[\sigma]})_u (g_{[0]})_y - (H_{[\sigma]})_y \big(g_{[0]} + mu(g_{[0]})_u\big) = u^r y^{s-1} B(t), \tag{8}$$

where

$$B(t) = -ptA(t)\Phi'(t) - sA(t)\Phi(t) - mq\sigma t A'(t)\Phi(t). \tag{9}$$

Indeed, the mixed $t^2 A'\Phi'$ terms cancel and $pr - qs = -q\sigma$ by (7).

Lemma 5.1 (Terminal coefficient) *Let*

$$A(t) = a_D t^D + \cdots, \qquad \Phi(t) = \phi_N t^N + \cdots,$$

with $D \geq 1$, $N \geq 0$, and $a_D \phi_N \neq 0$. Then

$$[t^{D+N}]B = -a_D \phi_N \big(pN + s + mq\sigma D\big) \neq 0. \tag{10}$$

Proof. The three summands in (9) contribute respectively

$$-pNa_D\phi_N, \qquad -sa_D\phi_N, \qquad -mq\sigma Da_D\phi_N.$$

The integer $pN + s + mq\sigma D$ is strictly positive because $s \geq 1$, $m, q, D \geq 1$, and $\sigma \geq 1$. It remains nonzero in k because $\operatorname{char} k = 0$. $\square$

Proof of 2.1. By 5.1, $B \neq 0$, so the Keller expression has a nonzero component of weight $\sigma - 1$.

If $\sigma > 1$, this is a positive-weight component, contradicting that the right side of (4) is the scalar c of weight zero.

If $\sigma = 1$, the entire top component in (8) must equal c. A scalar monomial can arise from $u^r y^{s-1} t^k$ only when $r = 0$, $s = 1$, and $k = 0$. Thus B would have to be constant. But 5.1 gives a nonzero term of degree $D + N \geq 1$, again a contradiction.

Therefore g is constant. By 3.1, that constant lies in $k^\times$. $\square$

Proof of 2.2. Write $g = a \in k^\times$ in (4). Then $-aH_y = c$, so

$$H(u, y) = -\frac{c}{a}y + h(u)$$

for some $h \in k[u]$. The inverse follows by solving $Q = ax$ and then solving the first coordinate for y. $\square$

6 Quotient identity and collisions

Define

$$K = ug^m.$$

A direct calculation gives

$$\operatorname{Jac}(H, K) = g^{m-1} \det JF. \tag{11}$$

Under $\det JF = c$ this yields

$$\operatorname{Jac}(H, K) = cg^{m-1}, \qquad u^{m-1} \operatorname{Jac}(H, K)^m = c^m K^{m-1}.$$

For $m \geq 2$, a hypothetical nonconstant g would also produce an explicit collision after passing to an algebraic closure. Since $g(0, y) \neq 0$, every zero (u_0, y_0) of g has $u_0 \neq 0$. Choosing $x_0^m = u_0$, the distinct points

$$(\zeta^j x_0, y_0), \qquad 0 \leq j < m,$$

all map under (1) to $(H(u_0, y_0), 0)$. The theorem rules out this collision mechanism. For $m = 1$ the rigidity theorem still holds, but this orbit-collision interpretation is vacuous.

7 Scope and the bridge at infinity

The theorem concerns the polynomial normal form (1) or, equivalently, the self-equivariant swapped form (2). It does not show that every hypothetical nonproper Keller map can be transformed into this form.

At a branch at infinity, tame inertia in residue characteristic zero is cyclic and acts on a local parameter by roots of unity. That one-dimensional local statement does not automatically provide a compatible second coordinate, simultaneous source and target linearization, or descent from completed local rings to global polynomial coordinates.

Nor does the terminal-coefficient argument extend unchanged to arbitrary completed series. For example, with $m = p = q = \sigma = s = 1$, $r = 0$, $A(t) = 1 + t$, and $\Phi(t) = (1 + t)^{-1} \in k[[t]]$, formula (9) gives

$$B(t) = -tA\Phi' - A\Phi - tA'\Phi = -1.$$

There is no terminal coefficient in an infinite power series. Any bridge from infinity must therefore prove a finite-support or bounded-pole statement, or replace the terminal coefficient by a different extremal invariant.

Question 7.1 *Under what verifiable hypotheses does a branch at infinity of a nonproper plane Keller map globalize to compatible polynomial coordinates carrying a cyclic pseudoreflection action?*

8 Reproducibility statement

The proof is algebraic and does not depend on finite search. The accompanying internal verification archive checks the determinant identity, the edge identity, the terminal coefficient over a large exact parameter grid, and bounded coefficient systems. A separate implementation performed 130 exact rational coefficient-system exclusions and 500 exact rational edge-identity substitutions. These calculations are regression evidence only; the unbounded conclusion is supplied by 5.1.

9 Novelty statement

A targeted literature audit located broad and important precedents in equivariant affine geometry, Newton polygons, geometry at infinity, and general Jacobian reductions, but no exact antecedent for 2.1 and 2.2. Accordingly, the calibrated claim is that the theorem and its direct terminal-edge proof appear new relative to the sources reviewed. A definitive priority claim requires specialist review.

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